Etude de systèmes spatialement étendus au voisinage de bifurcations

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Session Lille du GDR MOAD 21-23 mars 2007

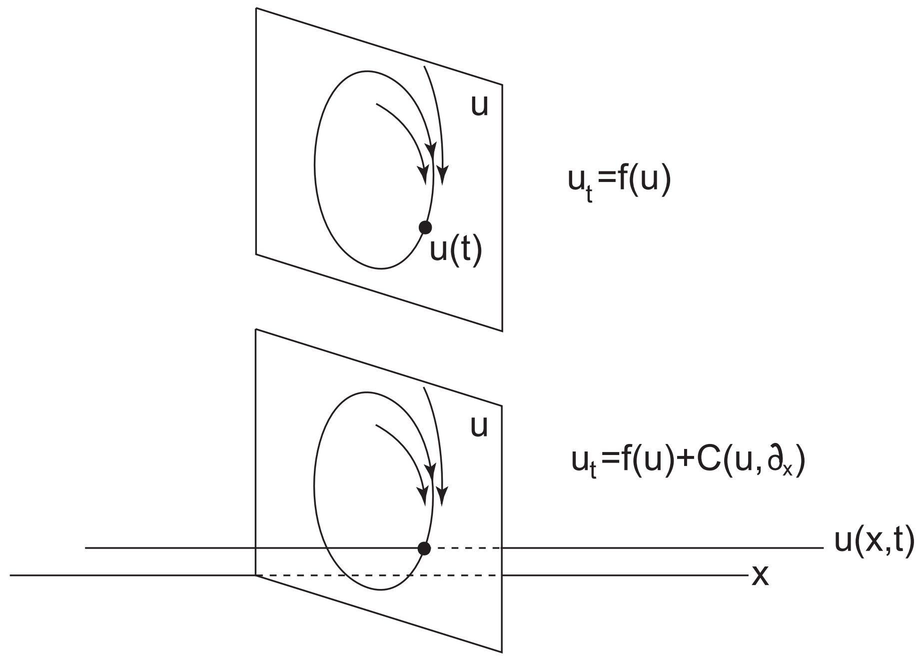
I. Déploiement spatial de la bifurcation homocline séparatrice à un nœud-col

II. Emission périodique de type I de pulses

I. Déploiement spatial de la bifurcation homocline séparatrice à un nœud-col

1. Cadre général
2. Déploiement spatial de bifurcation
3. Résultats

$$u_t = f(u) \quad \rightarrow \quad u_t = f(u) + \mathcal{C}(u, \partial_x), \quad \mathcal{C}(u, 0) = 0$$



Lien entre la dynamique de l'EDO et la dynamique de l'EDP (stabilité...)

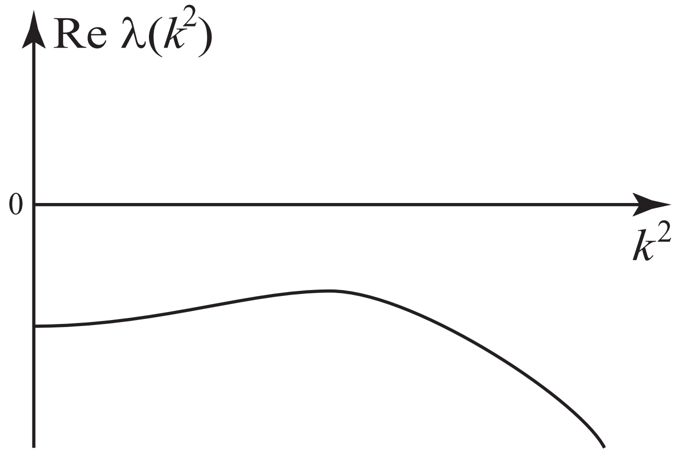
Exemples : -instabilité de Turing
-instabilité de phase de Kuramoto

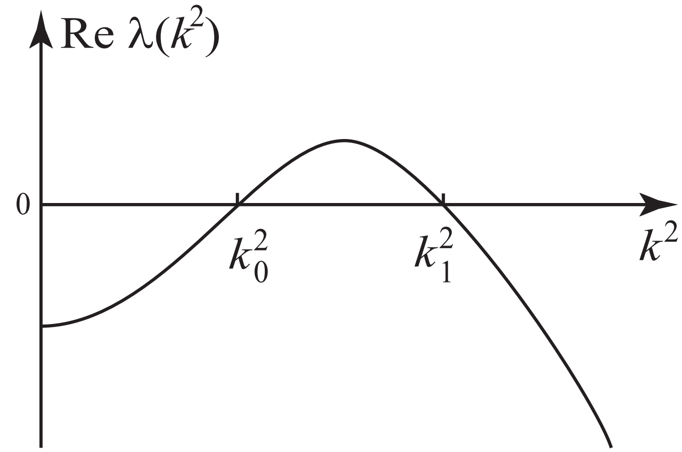
$u \in \mathbf{R}^2 \quad f(0) = 0 \quad \sigma(Df(0)) \subset \{\mathrm{Re}\, z < 0\}$

$$u_t = f(u) + Cu_{xx}$$

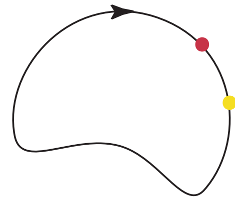
Equation linéarisée en 0 pour les modes de Fourier

$$\widehat{v}_t(k,t) = (Df(0) - k^2 C)\widehat{v}(k,t)$$

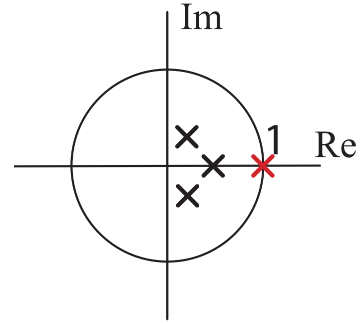




$t \mapsto p(t)$ solution périodique



Multiplicateurs de Floquet



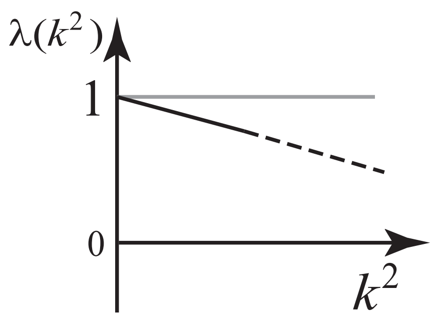
$u_t = f(u) + \mathcal{C}(u, \partial_x)$ espace isotrope

Pour k petit,

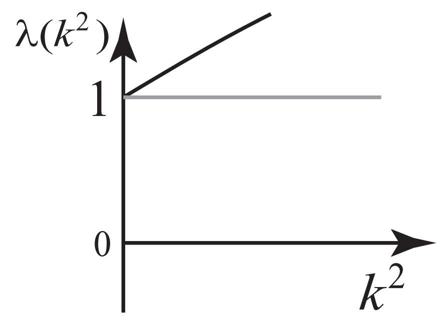
$$\hat{v}_t(k,t) = (Df(p(t)) - k^2 C(p(t)) + \mathcal{O}(k^4))\hat{v}(k,t)$$

$\Phi(k^2)$ opérateur de monodromie

$\partial_{k^2}\lambda(0) < 0$



$\partial_{k^2}\lambda(0) > 0$



I.2.Déploiement spatial de bifurcation

$$u_t = f(u, \mu)$$

Bifurcation en $\mu = 0$

$t \mapsto p_\mu(t)$ périodique linéairement stable, $\mu > 0$

Couplage spatial avec espace isotrope

Stabilité pour petits k

$$\widehat{v}_t = \Big(Df(p_\mu(t), \mu) - k^2 C(p_\mu(t), \mu) + \mathcal{O}(k^4) \Big) \widehat{v}$$

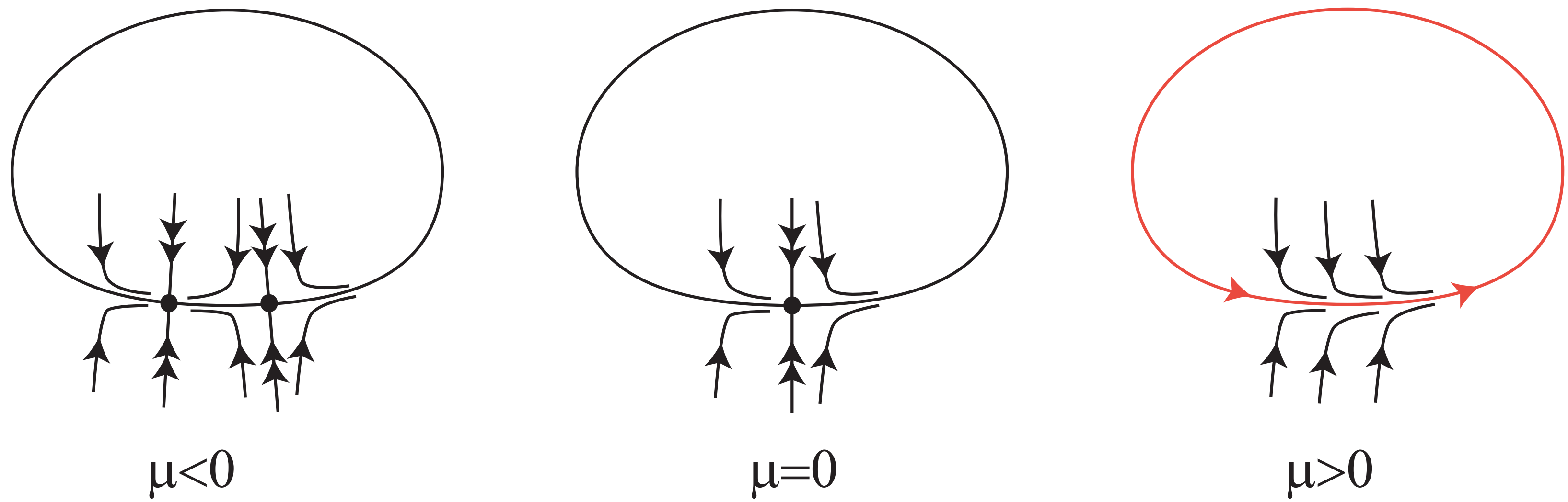
Opérateur de monodromie Φ_{μ,k^2}

Multiplicateur issu de la direction de la phase λ_{μ,k^2}

Argentina, Coullet, Risler, Vandenberghe (INLN)

I.2.1. Bifurcation homocline à un nœud-col

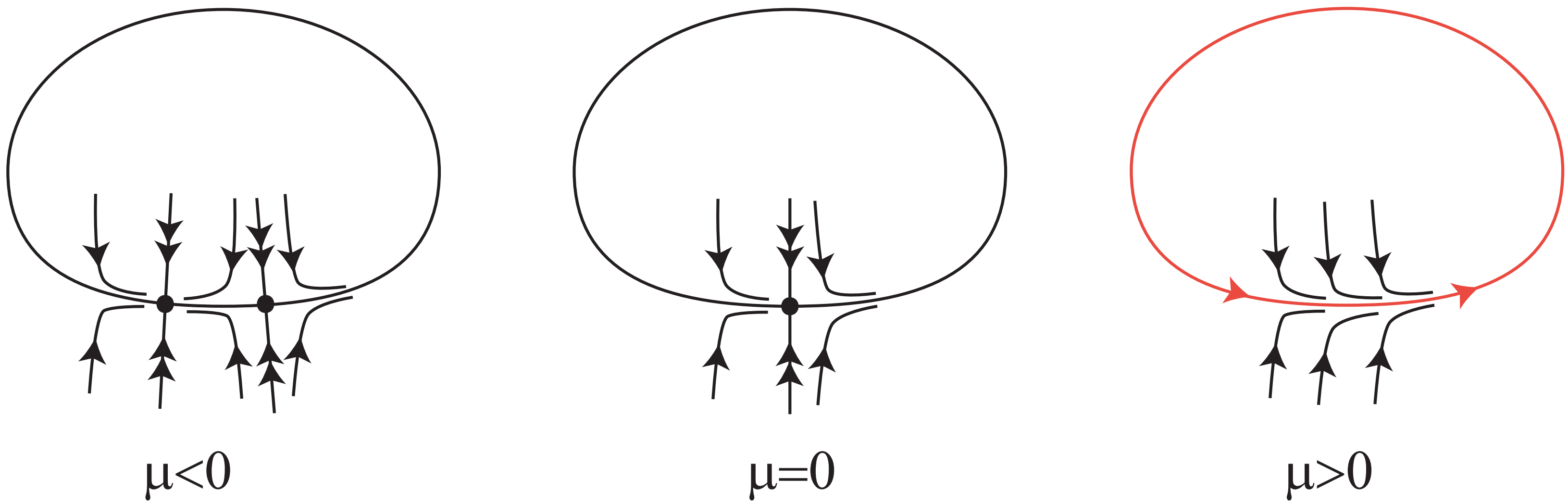
Nœud-col en $u = 0$, $\mu = 0$



$$C(0,0) = \begin{pmatrix} c & * \\ * & * \end{pmatrix} \qquad c = \langle e_0*, C(0,0)e_0 \rangle$$

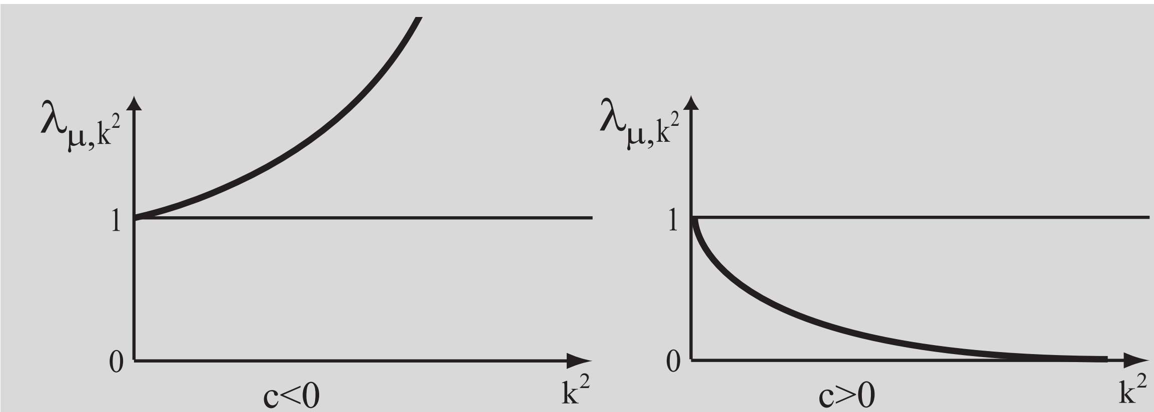
I.2.1. Bifurcation homocline à un nœud-col

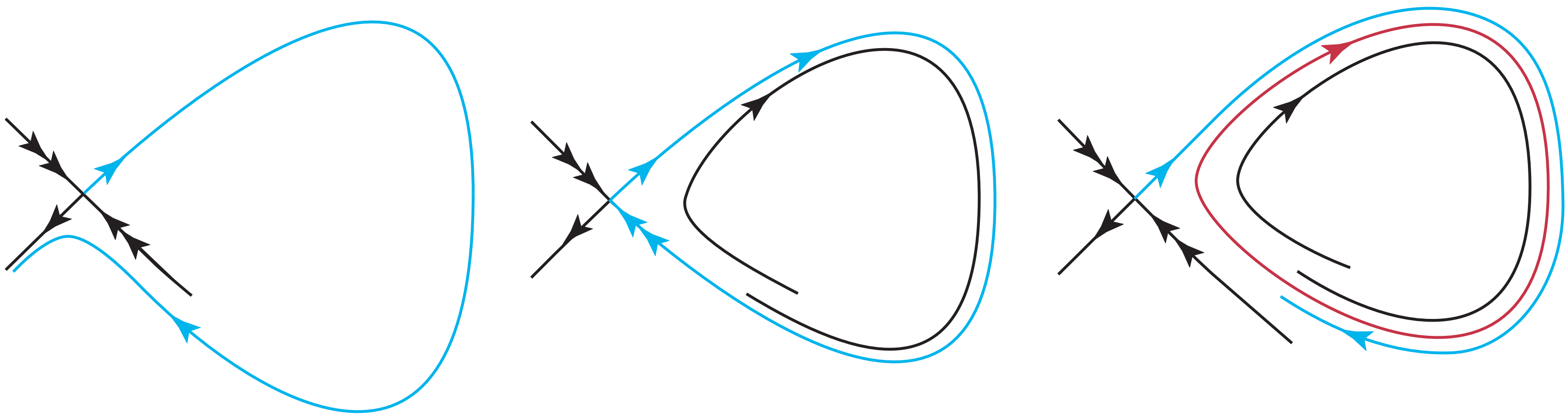
Nœud-col en $u = 0$, $\mu = 0$



$$C(0,0) = \begin{pmatrix} c & * \\ * & * \end{pmatrix} \qquad c = \langle e_0^*, C(0,0)e_0 \rangle$$

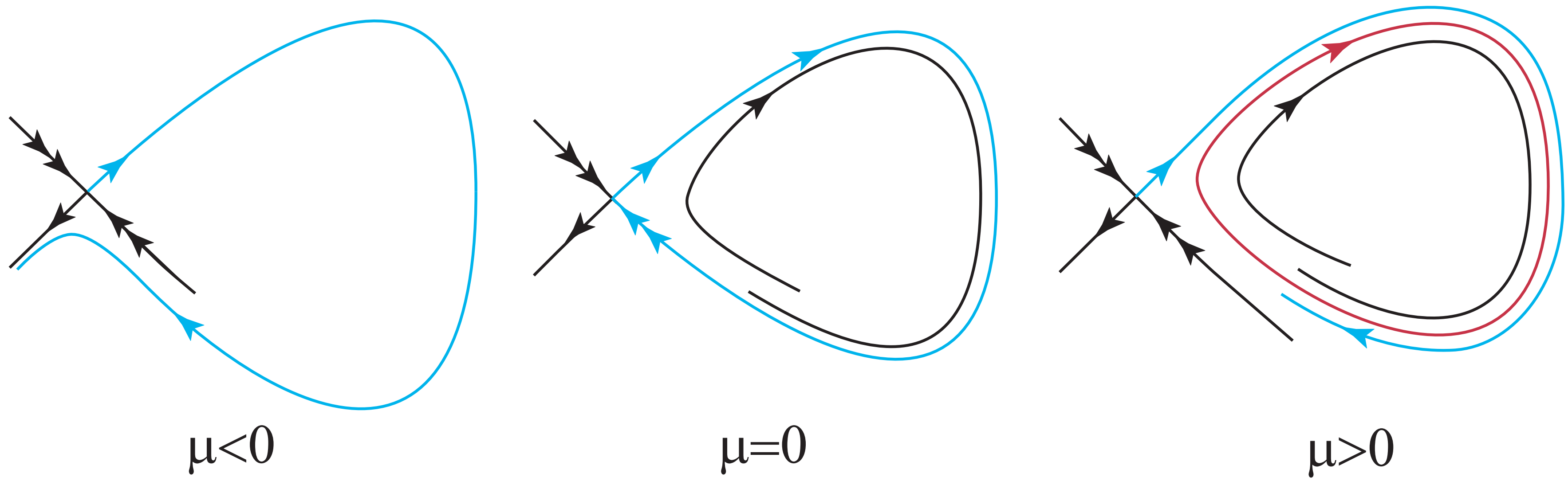
$$\lambda_{\mu,k^2} \simeq \exp(-cTk^2) \qquad T \propto \mu^{-1/2}$$





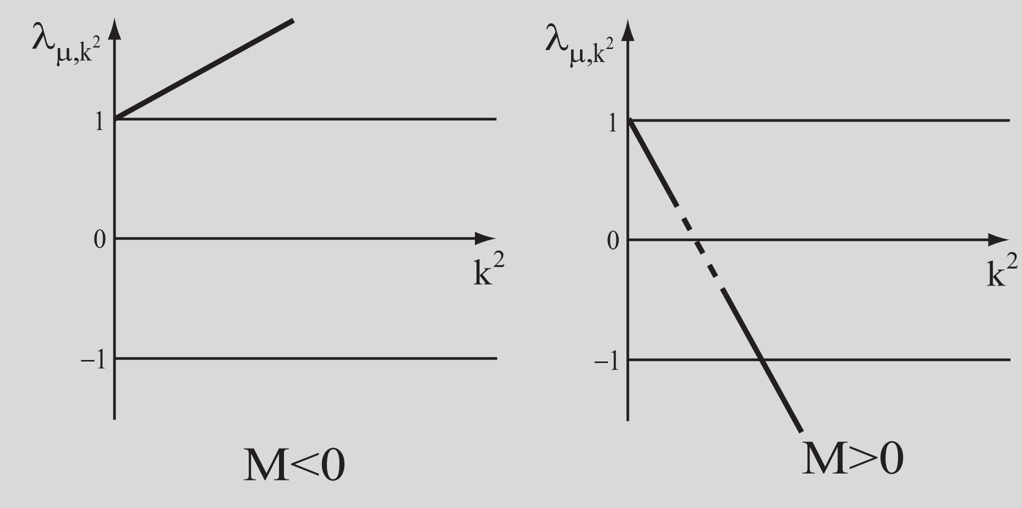
$\mu<0$ $\mu=0$ $\mu>0$

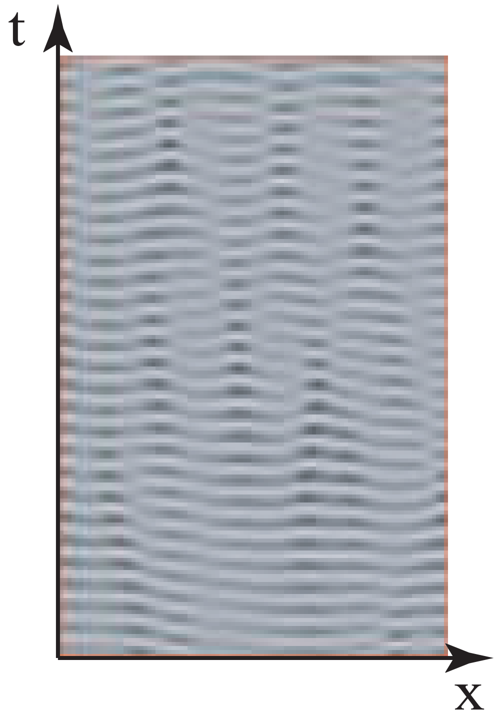
I.2.2 Bifurcation homocline à un point selle



$$M = \int_{-\infty}^{+\infty} \left\langle h_0^*(t), C(h_0(t), 0) h_0'(t) \right\rangle dt$$

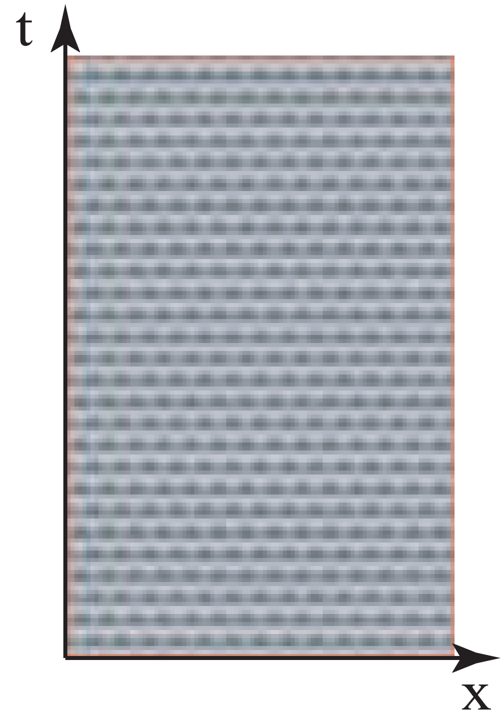
$$\lambda_{\mu,k^2} \simeq 1 - \alpha M k^2 \mu^{-1} \qquad \alpha > 0$$





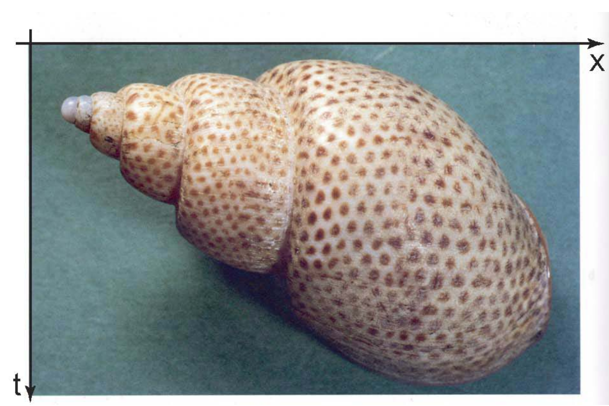
instabilité de phase

M<0



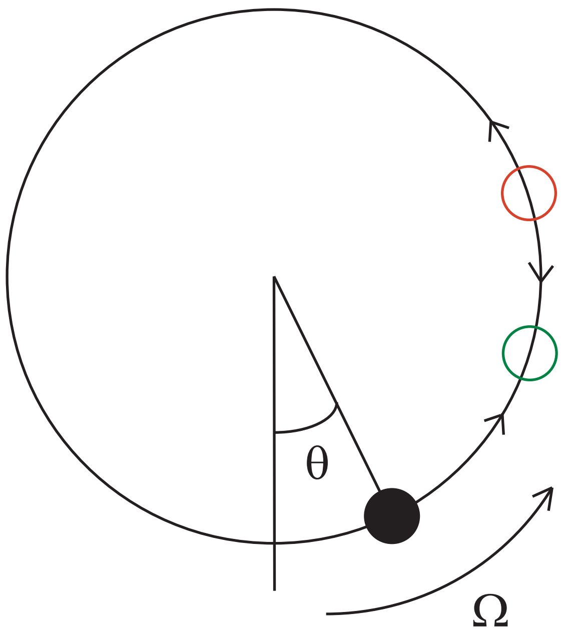
instabilité autoparamétrique

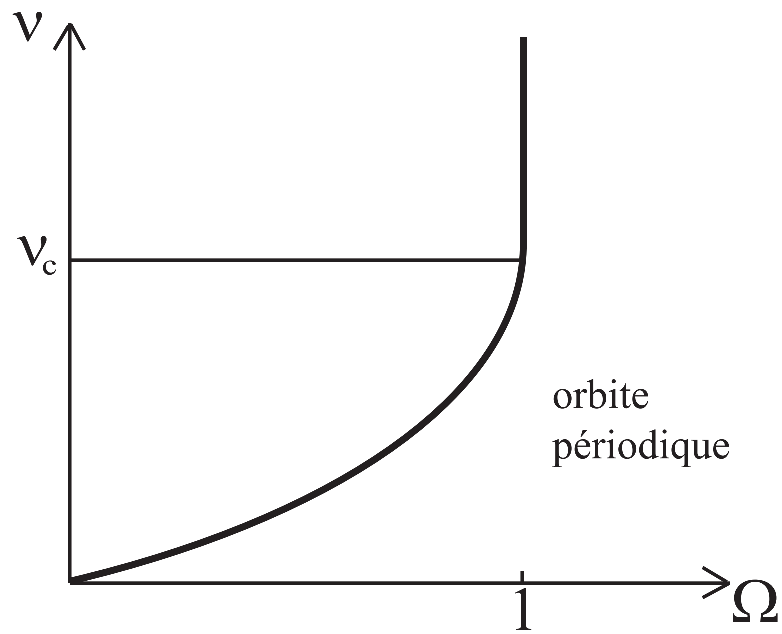
M>0



I.2.2.1. Exemple Pendule amorti soumis à un couple constant

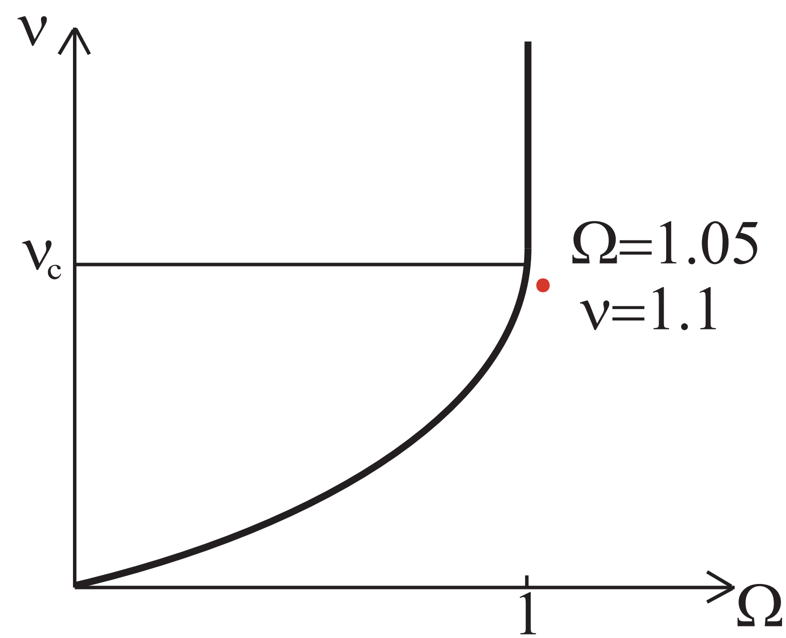
$$\ddot{\theta} = -\sin\theta - \nu\dot{\theta} + \Omega$$





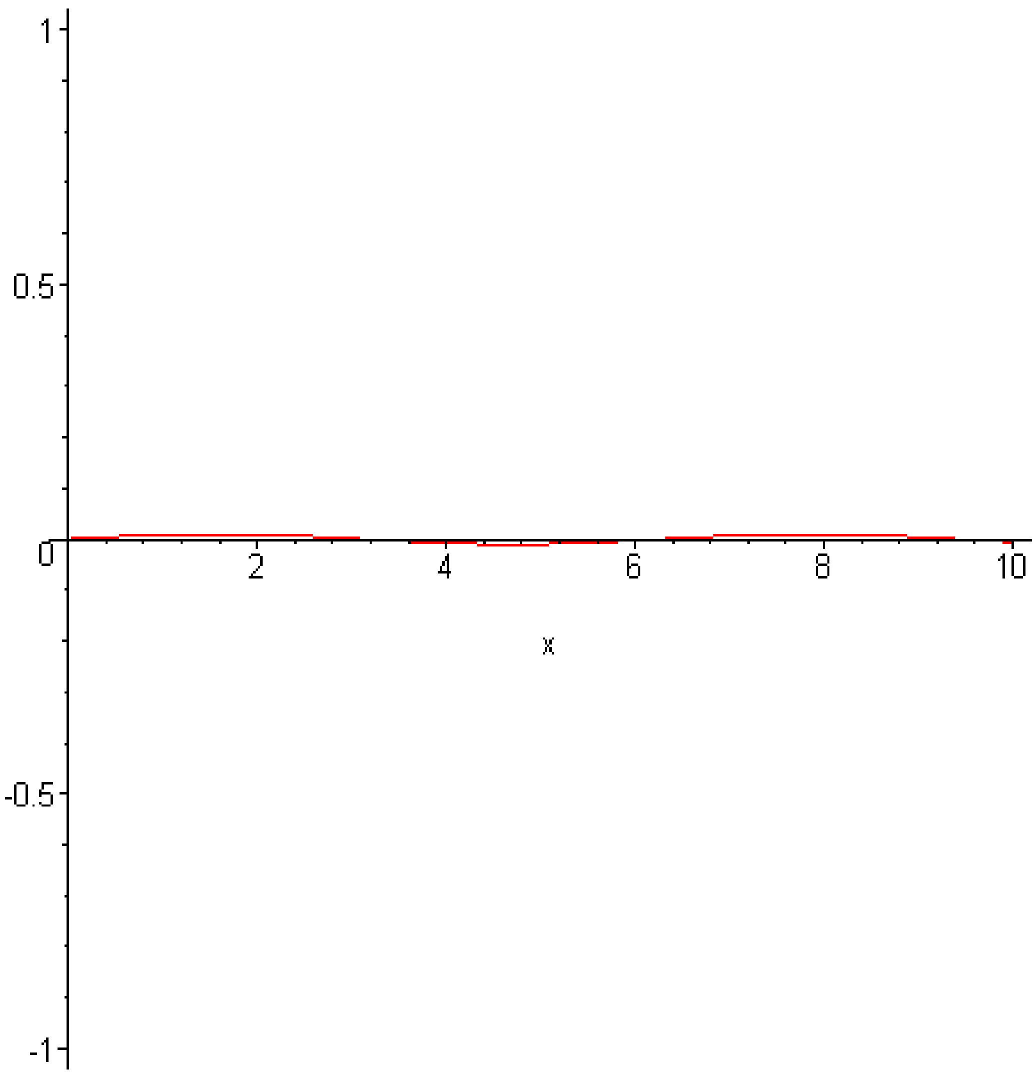
I.2.2.1.Exemple Pendule amorti soumis à un couple constant

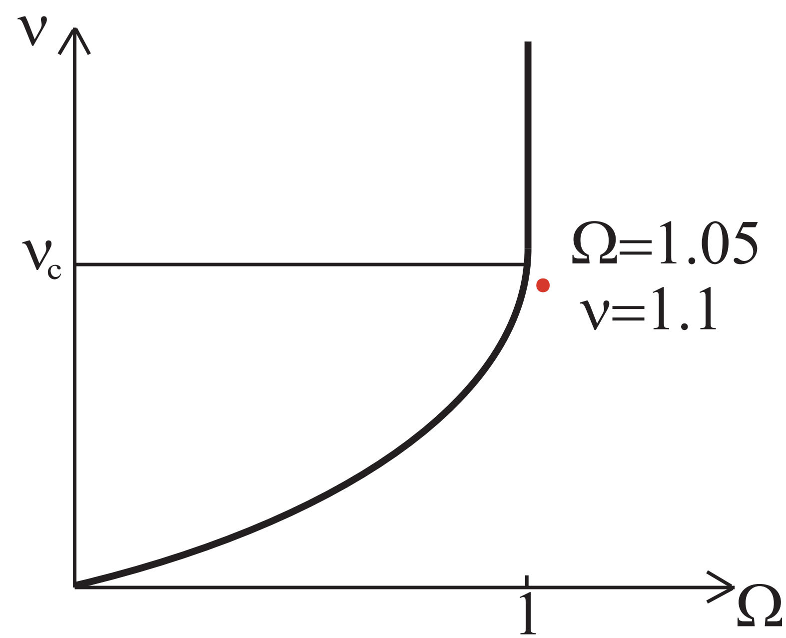
$$\ddot{\theta} = -\sin\theta - \nu\dot{\theta} + \Omega + \kappa\theta_{xx}$$



$\kappa = 0.044$

$$\ddot{\theta} = -\sin\theta - \nu\dot{\theta} + \Omega + \kappa\theta_{xx}$$

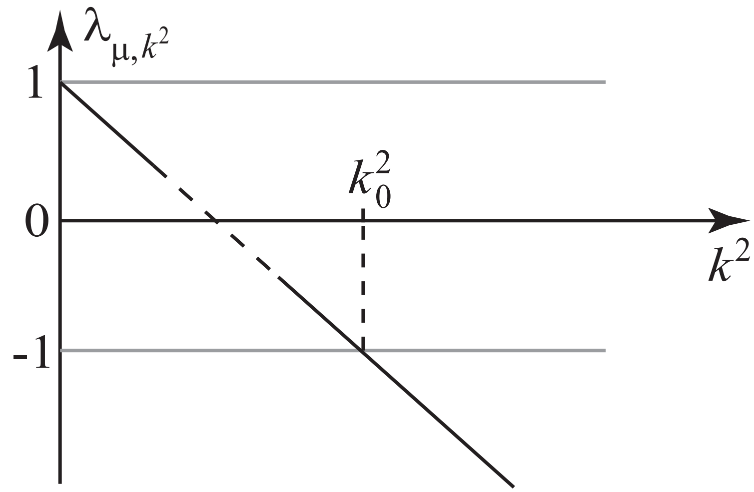




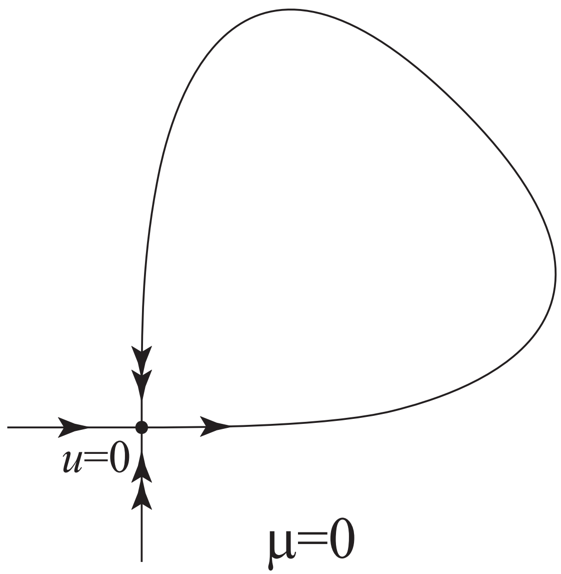
$\kappa = 0.044$

instabilité autoparamétrique $\quad M > 0$

$$\lambda_{\mu,k^2} \simeq 1 - \alpha M k^2 \mu^{-1}$$



I.3.Bifurcation homocline séparatrice à un noeud-col



$$u_t = f(u,\mu) + \mathcal{C}(u,\partial_x)$$

$$u = (u_1, u_2) \in \mathbf{R}^2$$
$$\mu = (\mu_1, \mu_2) \in \mathbf{R}^2$$

1.Etude de la bifurcation (*Chow et Lin*)

2.Influence du couplage spatial

$$\widehat{v}_t = \Big(Df(p_\mu(t),\mu) - k^2 C(p_\mu(t),\mu) + \mathcal{O}(k^4)\Big)\widehat{v}$$

Opérateur de monodromie Φ_{μ,k^2}

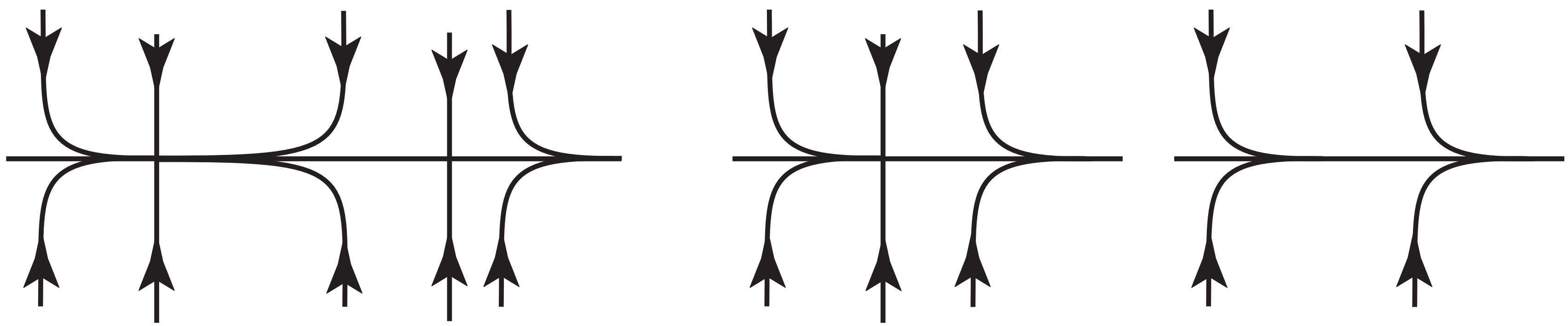
Multiplicateur issu de la direction de la phase λ_{μ,k^2}

3.Etude non-linéaire au seuil

$$M > 0, c > 0$$

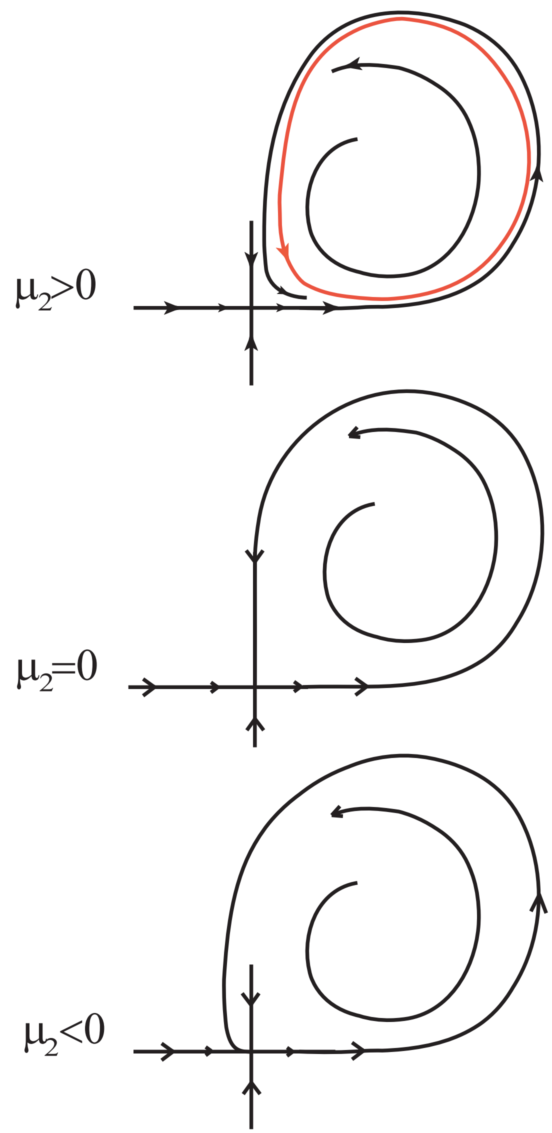
Au voisinage de 0,

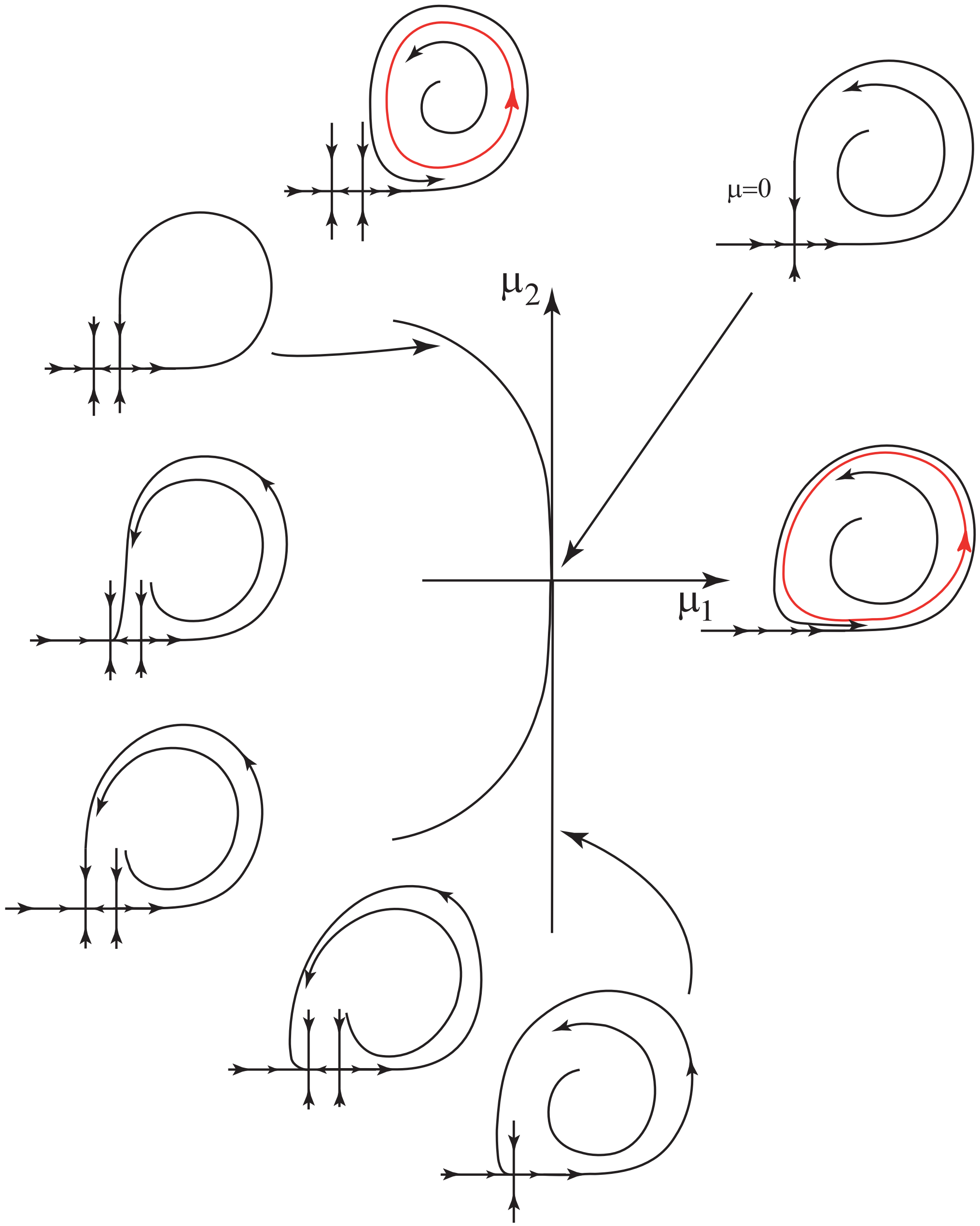
$$\begin{cases} \dot{u}_1 = f_1(u,\mu) \simeq \mu_1 + u_1^2 \\ \dot{u}_2 = f_2(u,\mu) \simeq -u_2 \end{cases}$$

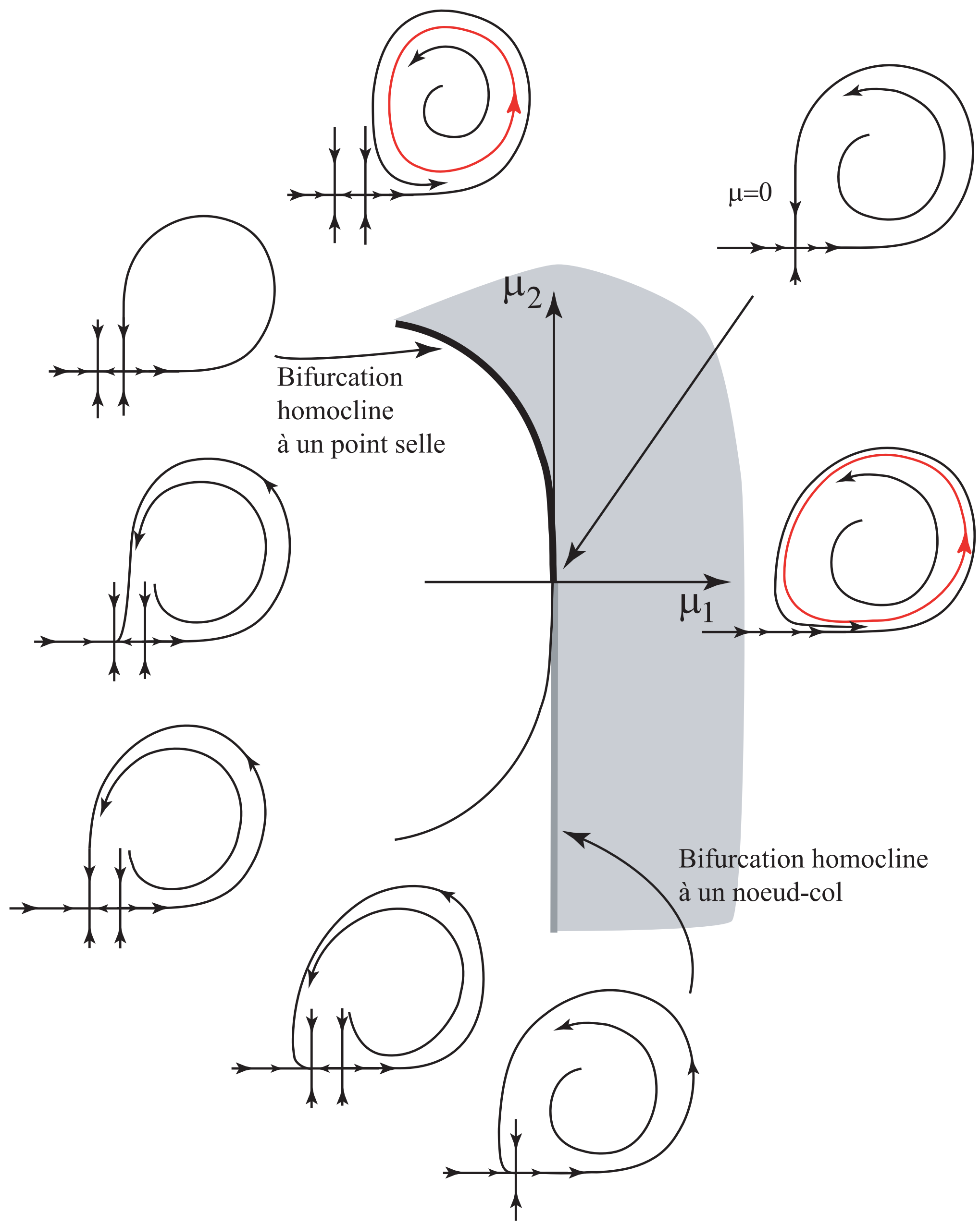


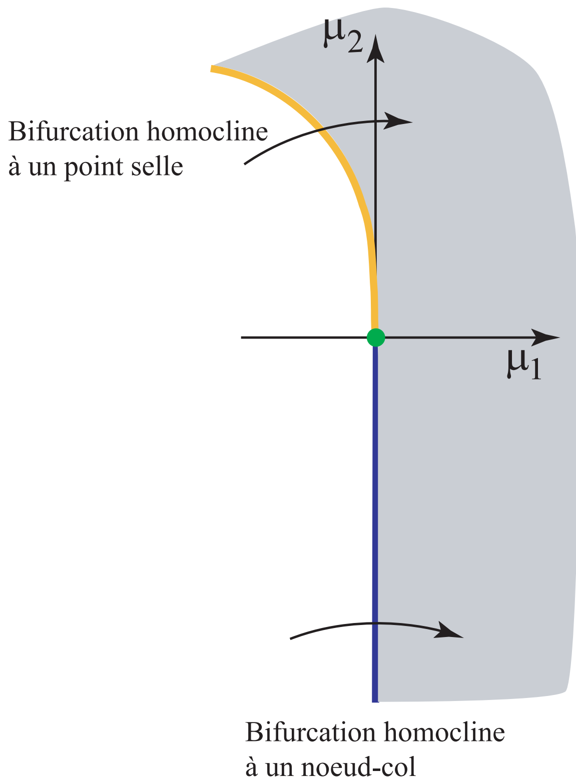
$\mu_1<0$ $\mu_1=0$ $\mu_1>0$

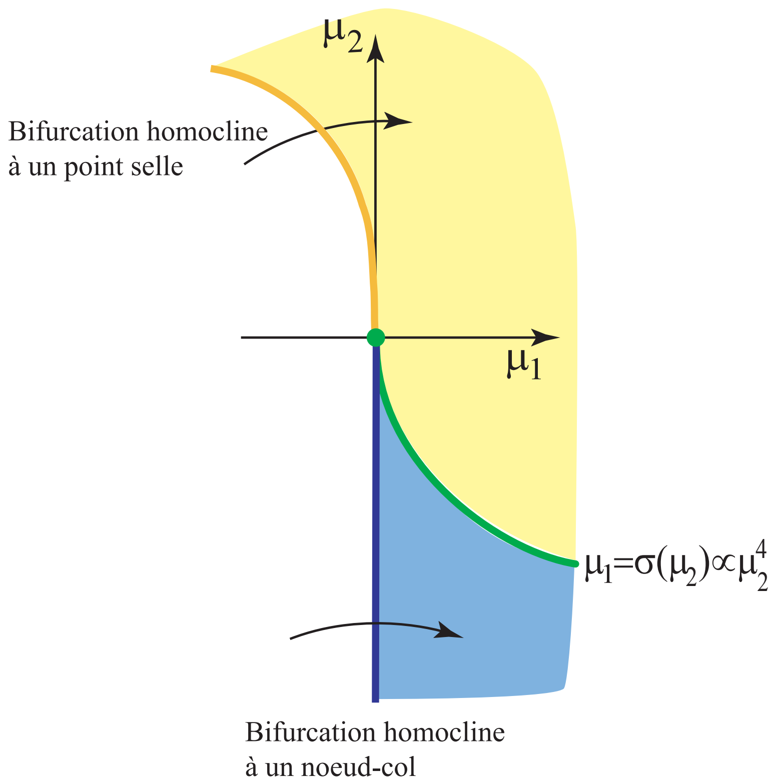
pour $\mu_1 = 0$



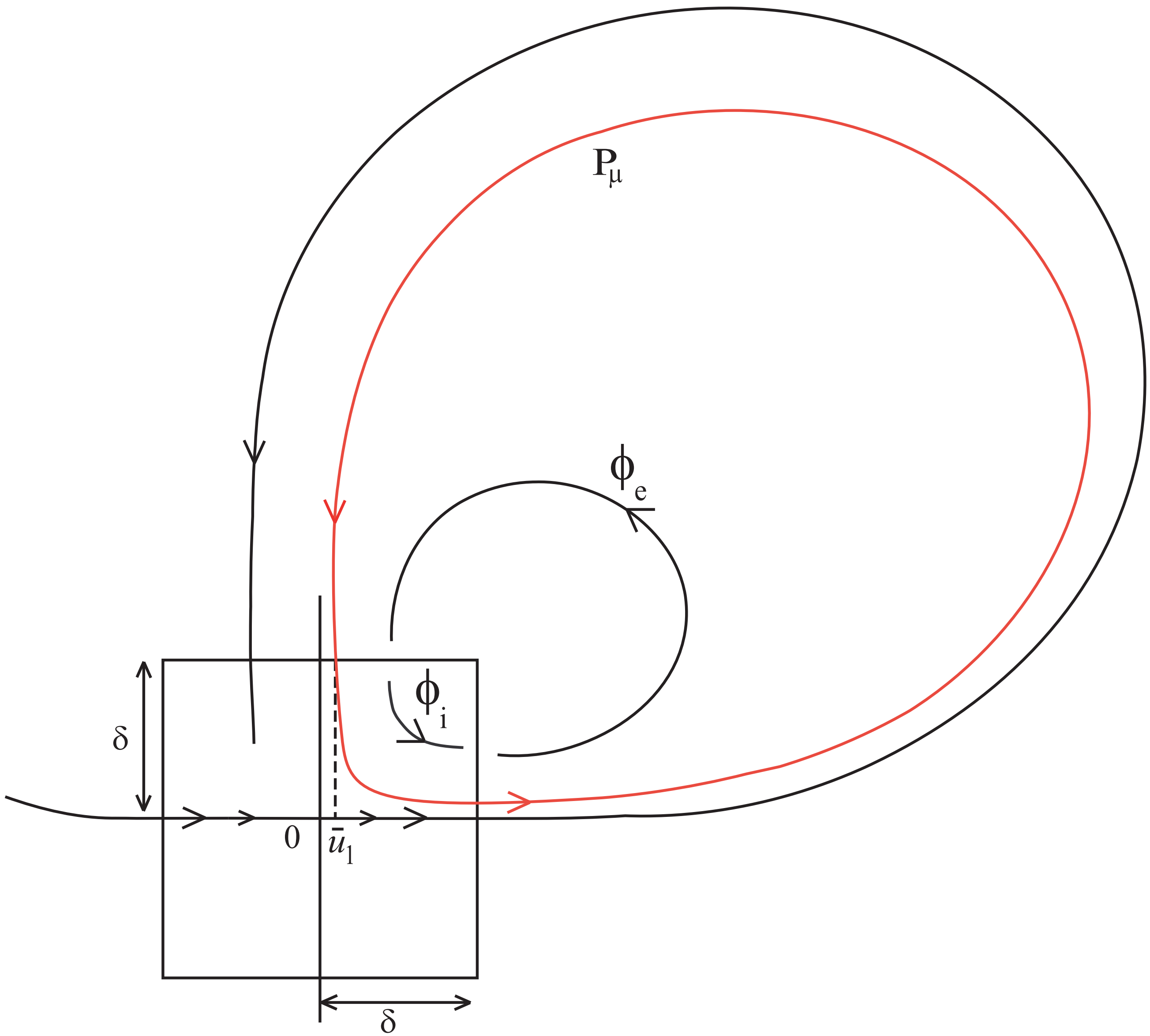

μ=0
μ_2
μ_1


μ=0
μ2
μ1
Bifurcation
homocline
à un point selle
Bifurcation homocline
à un noeud-col


μ_2
Bifurcation homocline
à un point selle
μ_1
Bifurcation homocline
à un noeud-col


μ_2
Bifurcation homocline
à un point selle
μ_1
$\mu_1 = \sigma(\mu_2) \propto \mu_2^4$
Bifurcation homocline
à un noeud-col

$$\mu_1 > 0$$

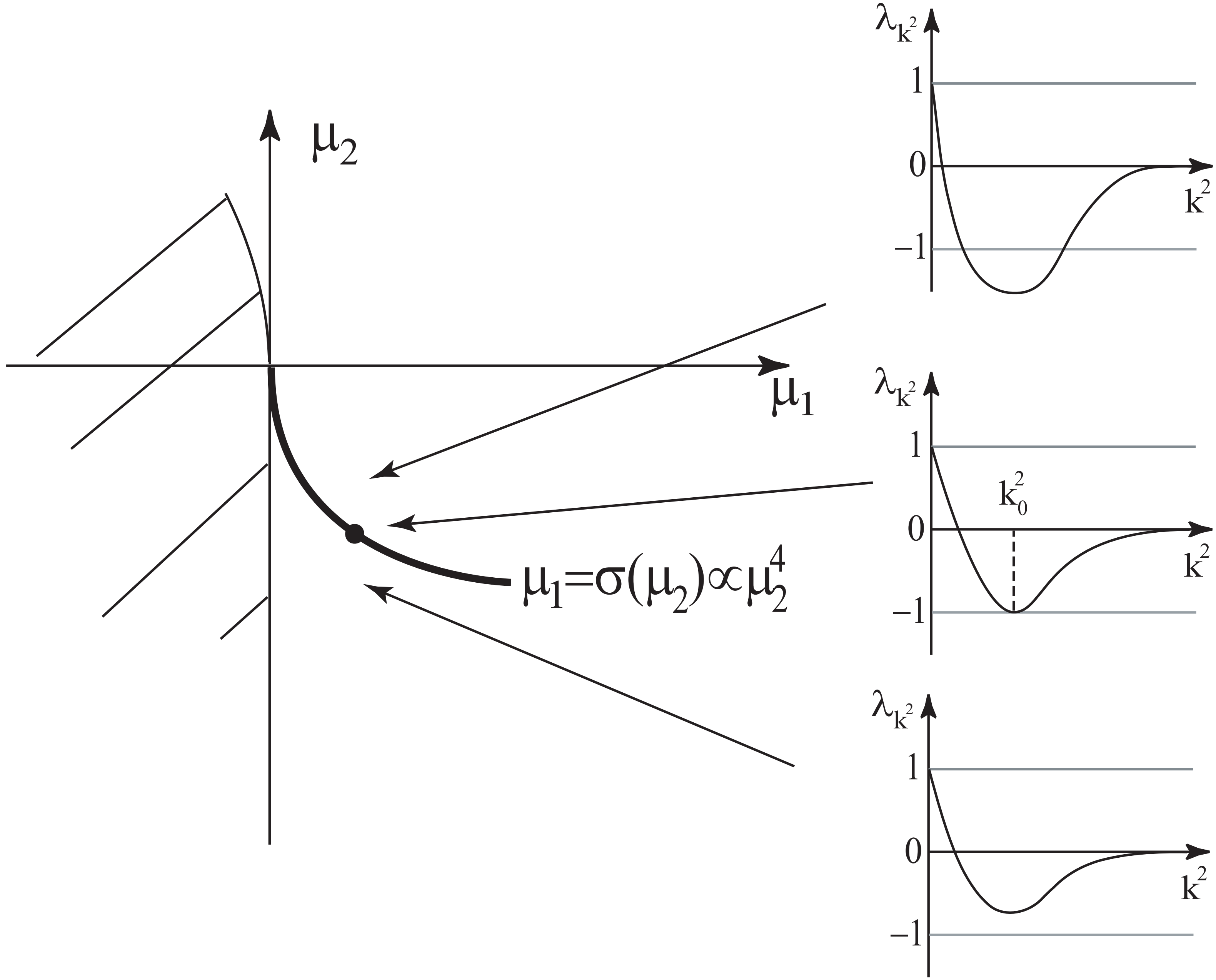


Hypothèses (H) sur la transversalité de la bifurcation et les orientations des variables et paramètres.

Théorème *Si f et C vérifient (H), alors il existe $\alpha > 0$ tel que pour μ et k^2 suffisamment petits, Φ_{μ,k^2} a une valeur propre proche de 0 et une valeur propre $\lambda(\mu, k^2)$ telle que*

$$\lambda(\mu, k^2) \simeq (1 - \alpha M f_1(\bar{u}_1)^{-1} k^2) \exp(-cTk^2).$$

Proposition $T \propto f_1(\bar{u}_1)^{-1} \iff \mu_2 \propto \mu_1^4\ , \quad \mu_2 < 0$



μ_2
μ_1
$\mu_1=\sigma(\mu_2)\propto\mu_2^4$
λ_{k^2}
1
0
−1
k^2
λ_{k^2}
1
k_0^2
0
−1
k^2
λ_{k^2}
1
0
−1
k^2

Dynamique sur la variété centrale

$$u(\tau) = p_\mu(\tau) + Az(\tau)e^{ik_0x} + \bar{A}z(\tau)e^{-ik_0x} + \phi(\tau, A, \bar{A}, \mu),$$

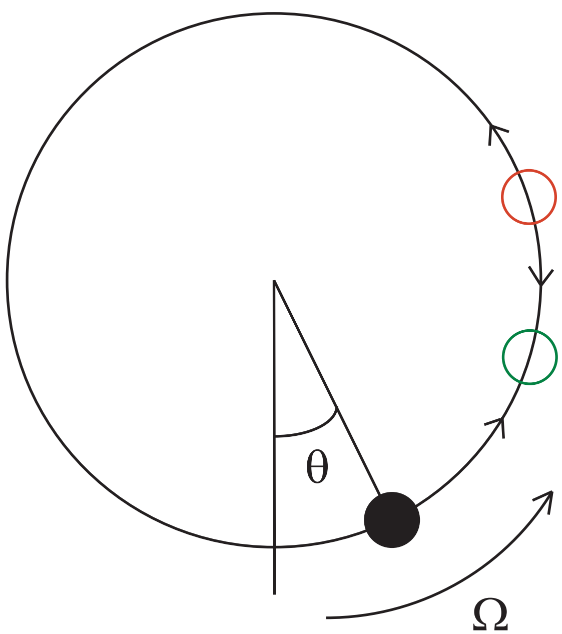
$$\partial_\tau A = c(\mu - \mu_0)A + d|A|^2 A + ...$$
$$\partial_t \tau = 1 + p(\tau, |A|^2, \mu) + ...$$

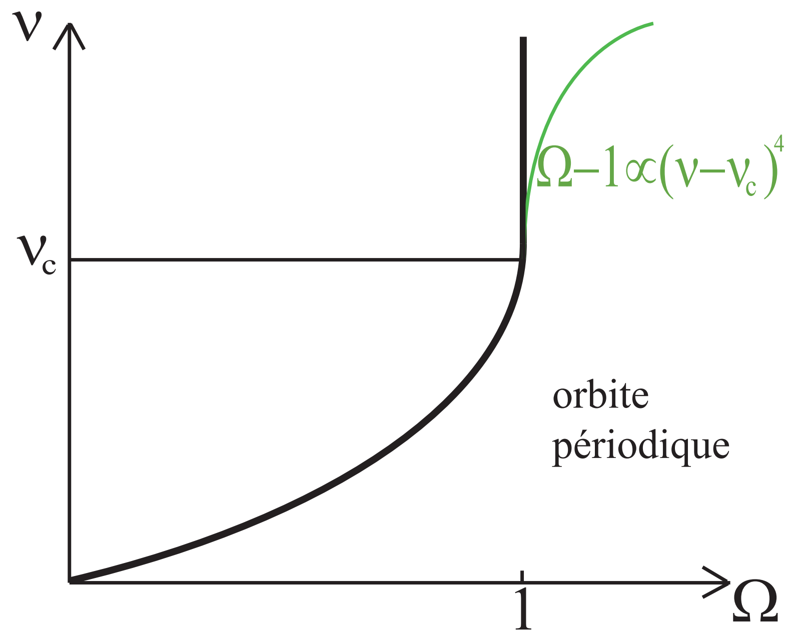
$$N_0 = \int_{-\infty}^{+\infty} \left\langle h_0^*(t), D^3 f(h_0(t), 0)(h_0'(t)^{(3)}) \right\rangle dt$$

Propriété *Si* $N_0 > 0$*, alors l'instabilité est supercritique.*
Si $N_0 < 0$*, l'instabilité est sous-critique.*

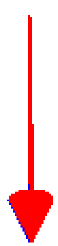
I.3.4.Exemple Pendule amorti soumis à un couple constant

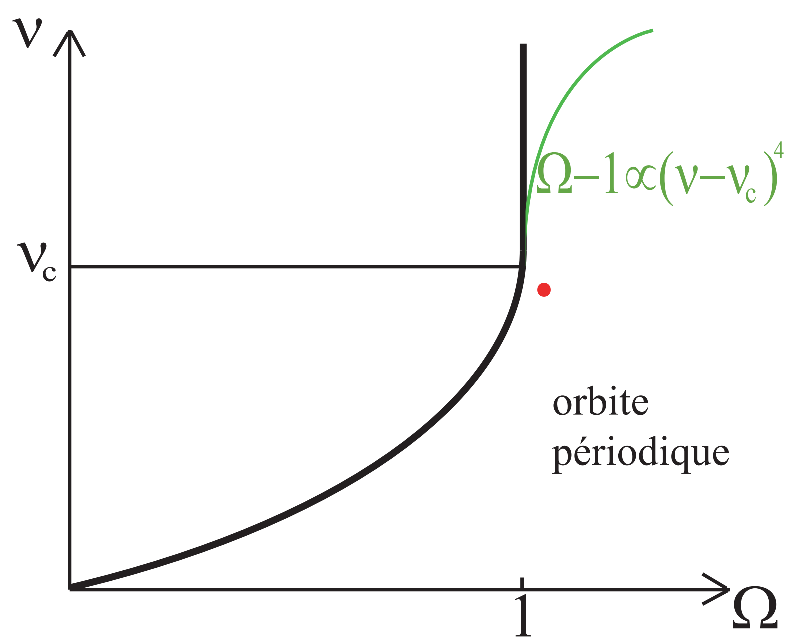
$$\ddot{\theta} = -\sin\theta - \nu\dot{\theta} + \Omega + \theta_{xx}$$
$$c > 0, M > 0$$
$$N_0 \simeq 11.8$$





$$\ddot{\theta}_1 = -\sin\theta_1 - \nu\dot{\theta}_1 + \Omega + \kappa(\theta_2 - \theta_1)$$
$$\ddot{\theta}_2 = -\sin\theta_2 - \nu\dot{\theta}_2 + \Omega + \kappa(\theta_1 - \theta_2)$$
$$c > 0, M > 0$$
$$N_0 \simeq 11.8$$





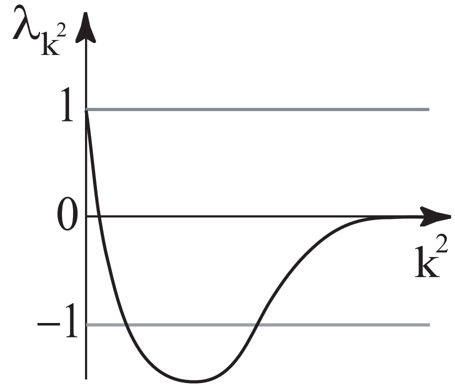
$$\Omega = 1.05, \nu = 1.1, \kappa = 0.1$$

I.3.5. Perspectives

Etude non-linéaire au seuil en domaine d'espace borné

↓

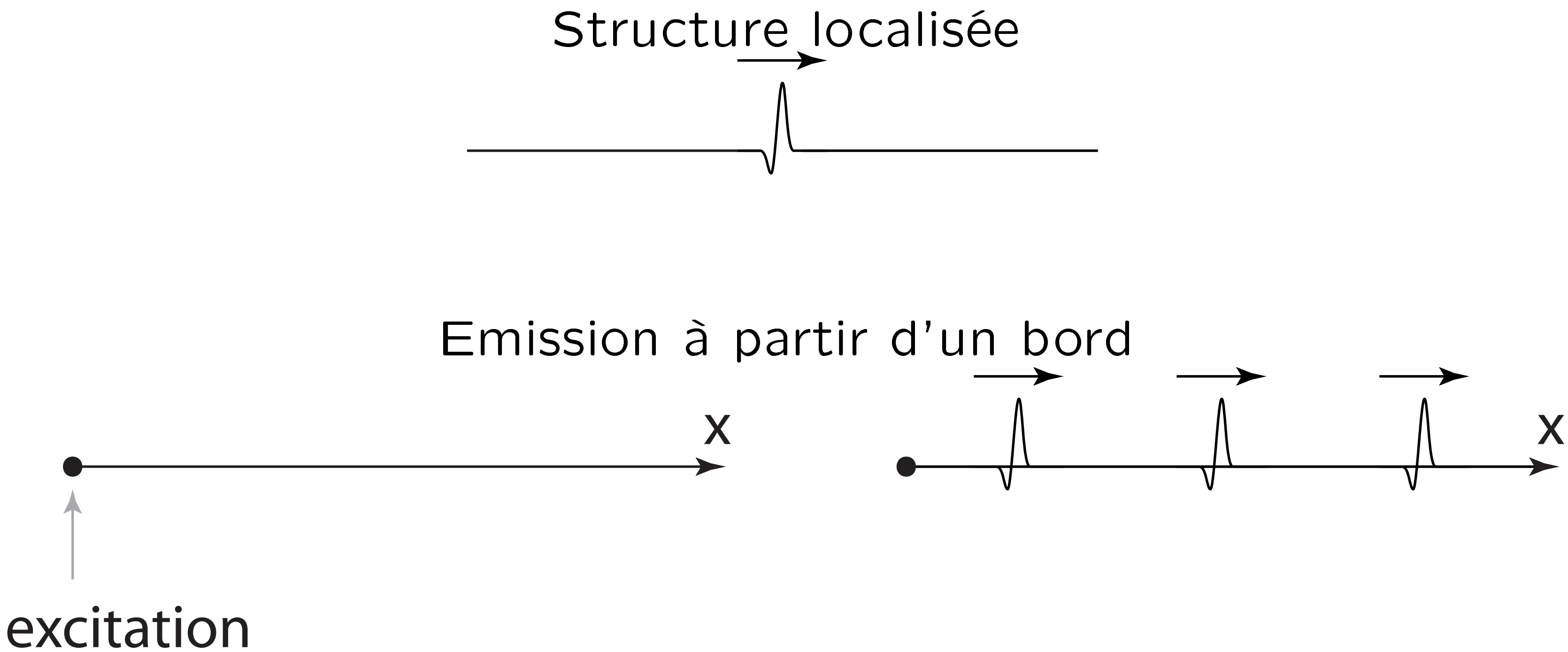
domaine non borné



Variété centrale de dimension infinie *Schneider*

II. Emission périodique de type I de pulses

II.1.Emission de structures localisées

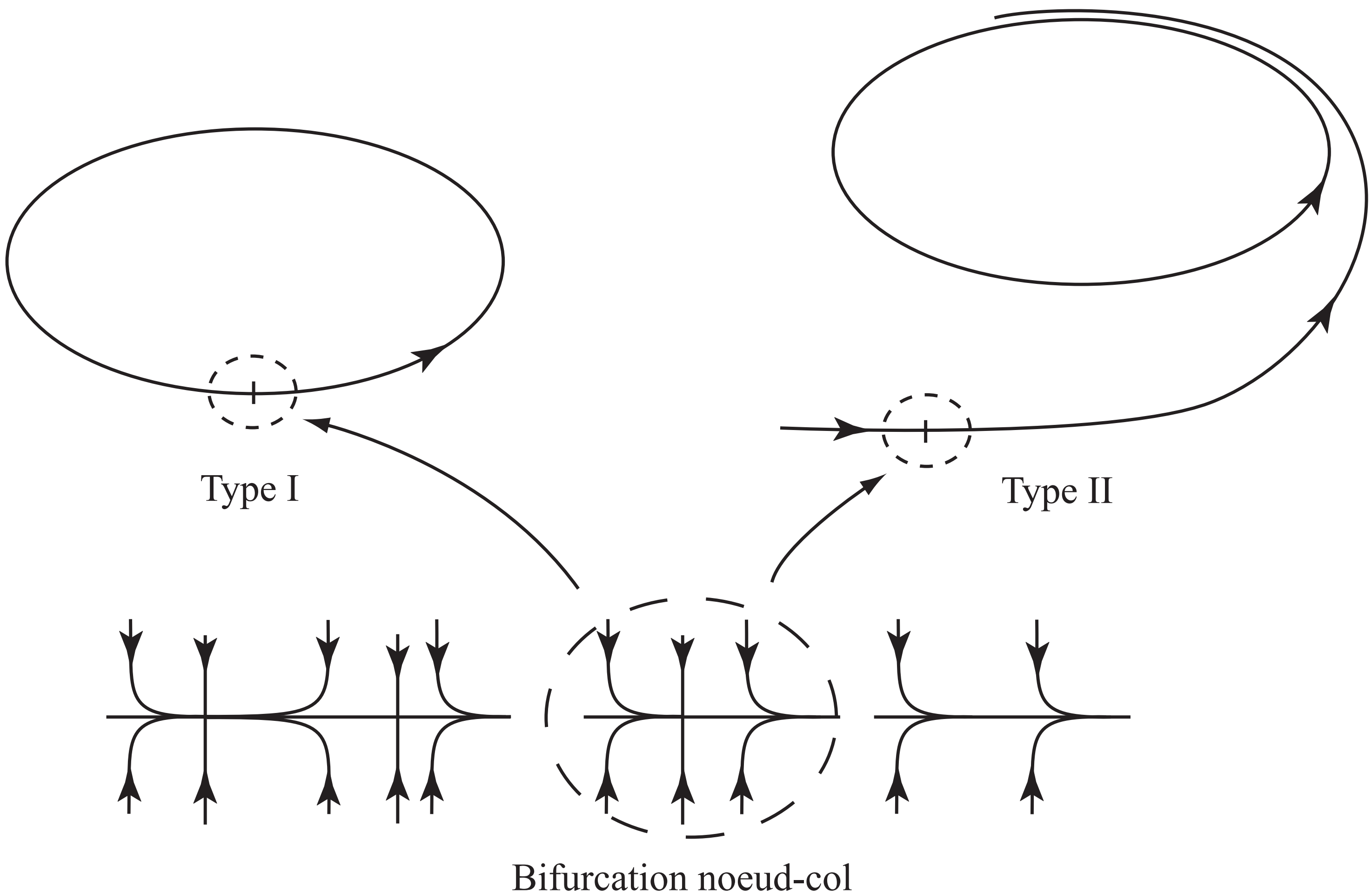


2 types d'émission périodique. Au seuil :

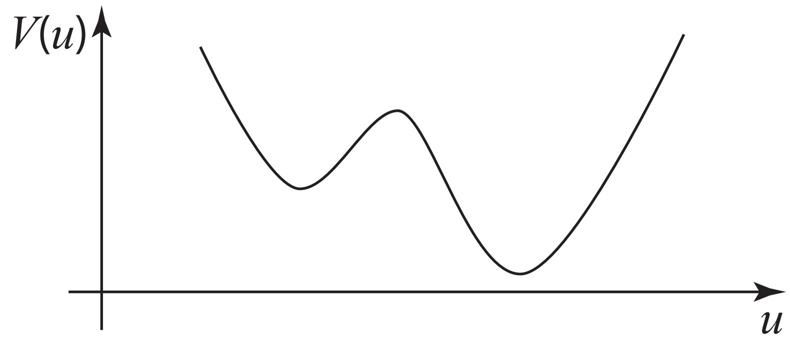
– fréquence nulle $\longrightarrow$ type I

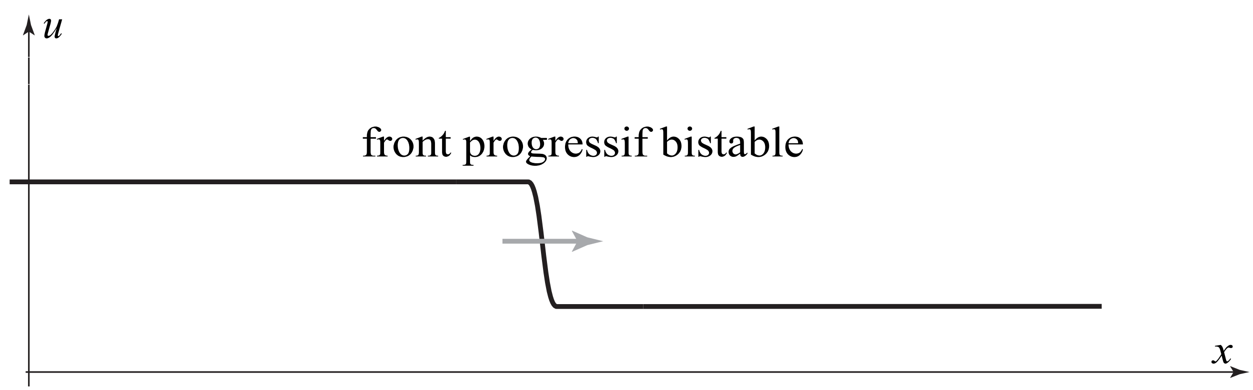
– fréquence finie $\longrightarrow$ type II

Dynamique dans l'EDP



$$u_t = -V'(u) + u_{xx}, x \in \mathbf{R}$$

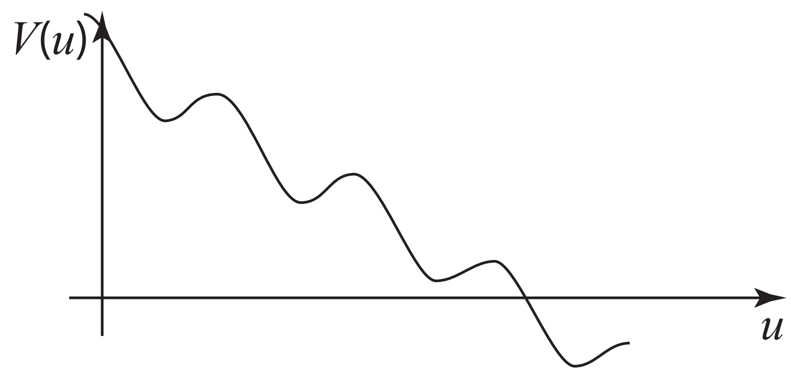




$$u_t = -\sin u + \Omega + u_{xx} = -V'(u) + u_{xx}$$

$x \in \mathbf{R}$, $0 < \Omega < 1$

Sine-Gordon forcée sans inertie



Principe du maximum

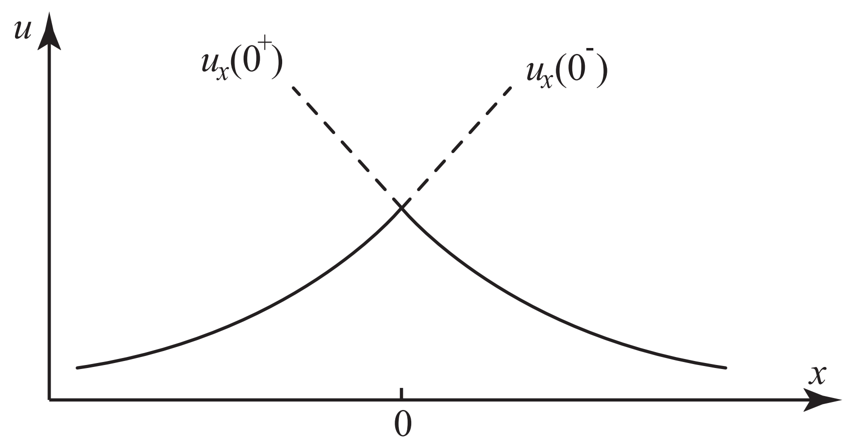
Fonctionnelle de Lyapunov

Inhomogénéité ponctuelle

$$u_t = -\sin u + \Omega + u_{xx} + \nu\delta_0 \qquad x \in \mathbf{R} \qquad \nu > 0$$

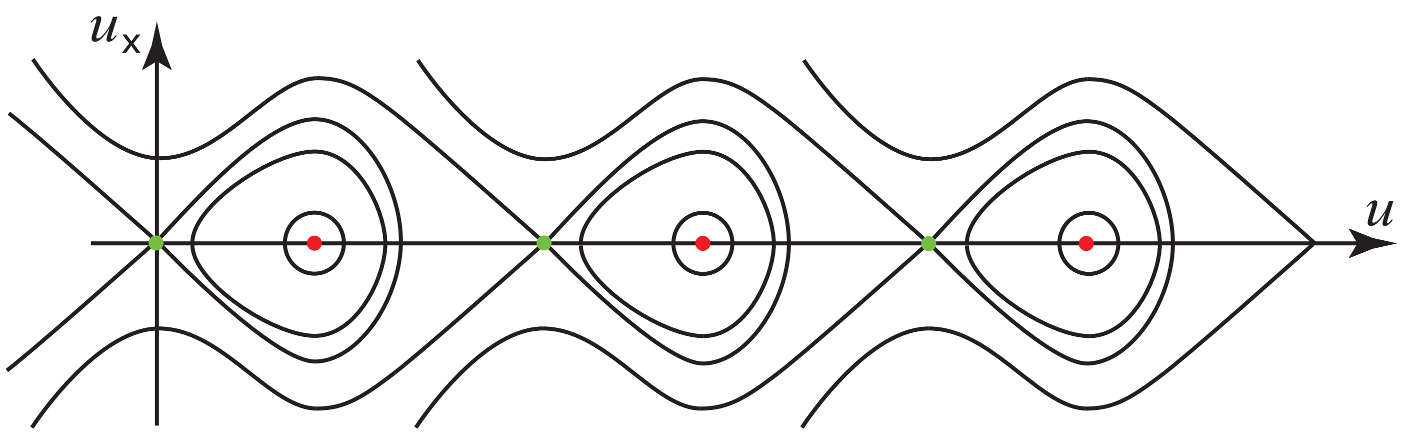
Solutions paires

$$\begin{cases} u_t = -\sin u + \Omega + u_{xx} & x \geq 0 \\ u_x(x=0,t) = -\phi & \phi = \nu/2 > 0 \end{cases}$$



Espace des phases des solutions stationnaires

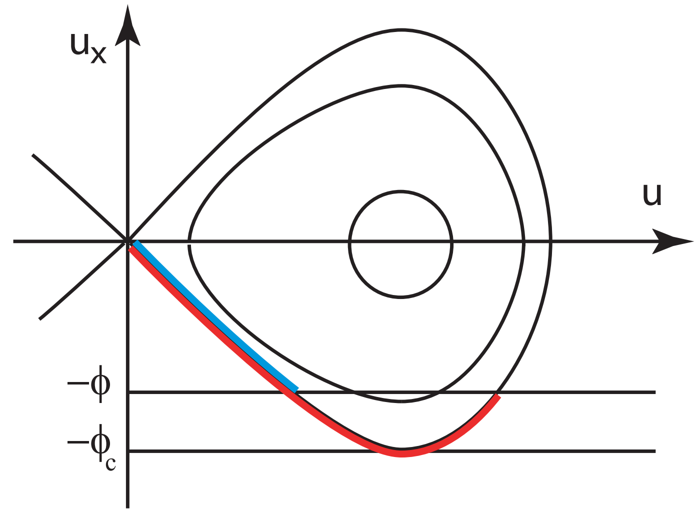
$$u_{xx} = \sin u - \Omega$$
$$0 < \Omega < 1$$
$$x \in \mathbf{R}$$

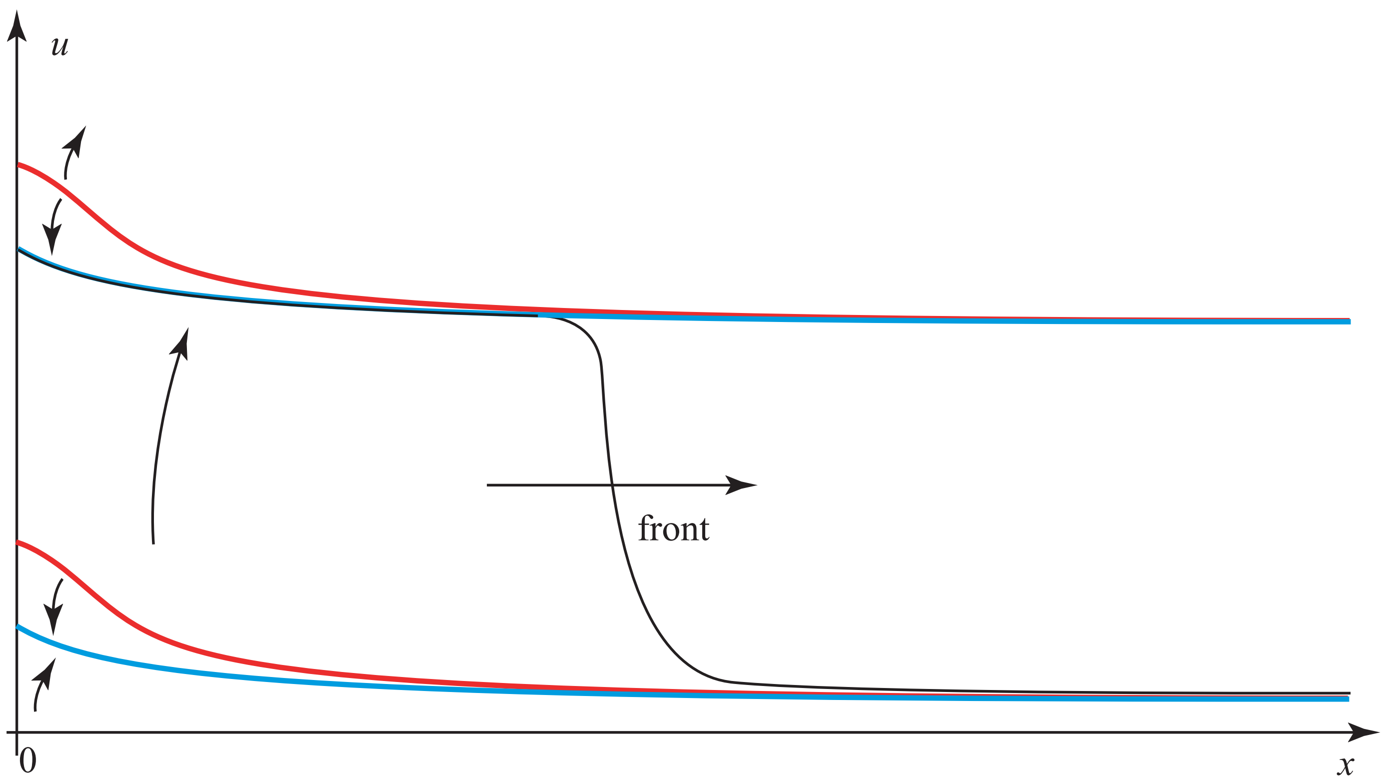


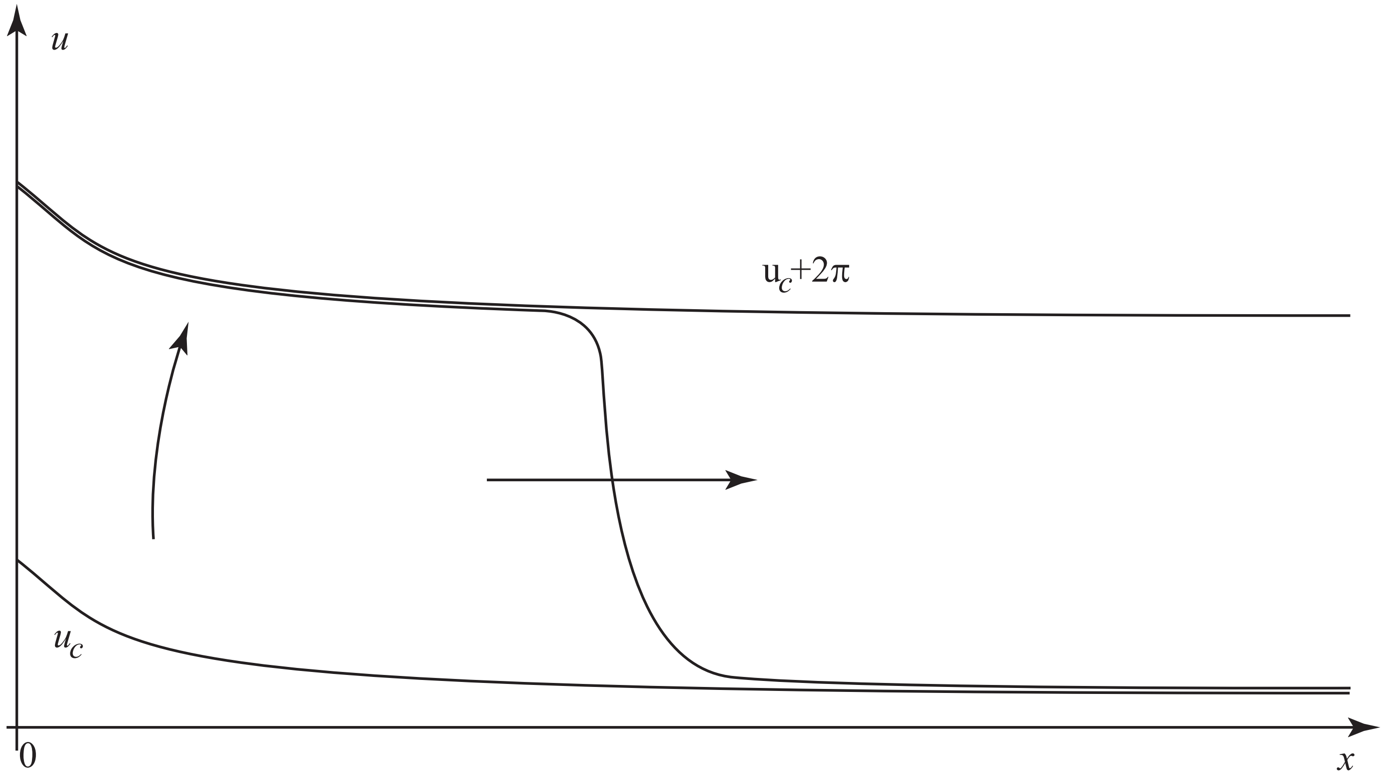
$x \in \mathbf{R}_+$

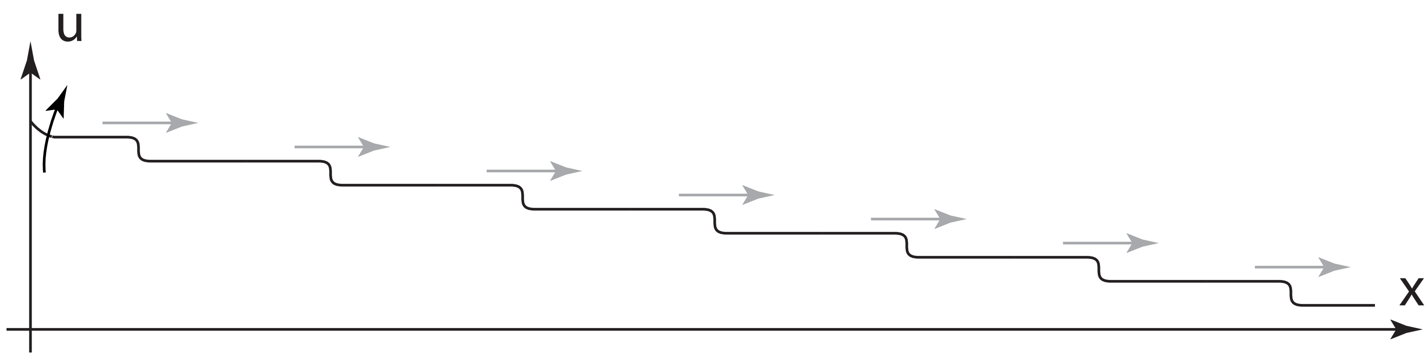
Neumann non homogène

Stabilité à l'infini





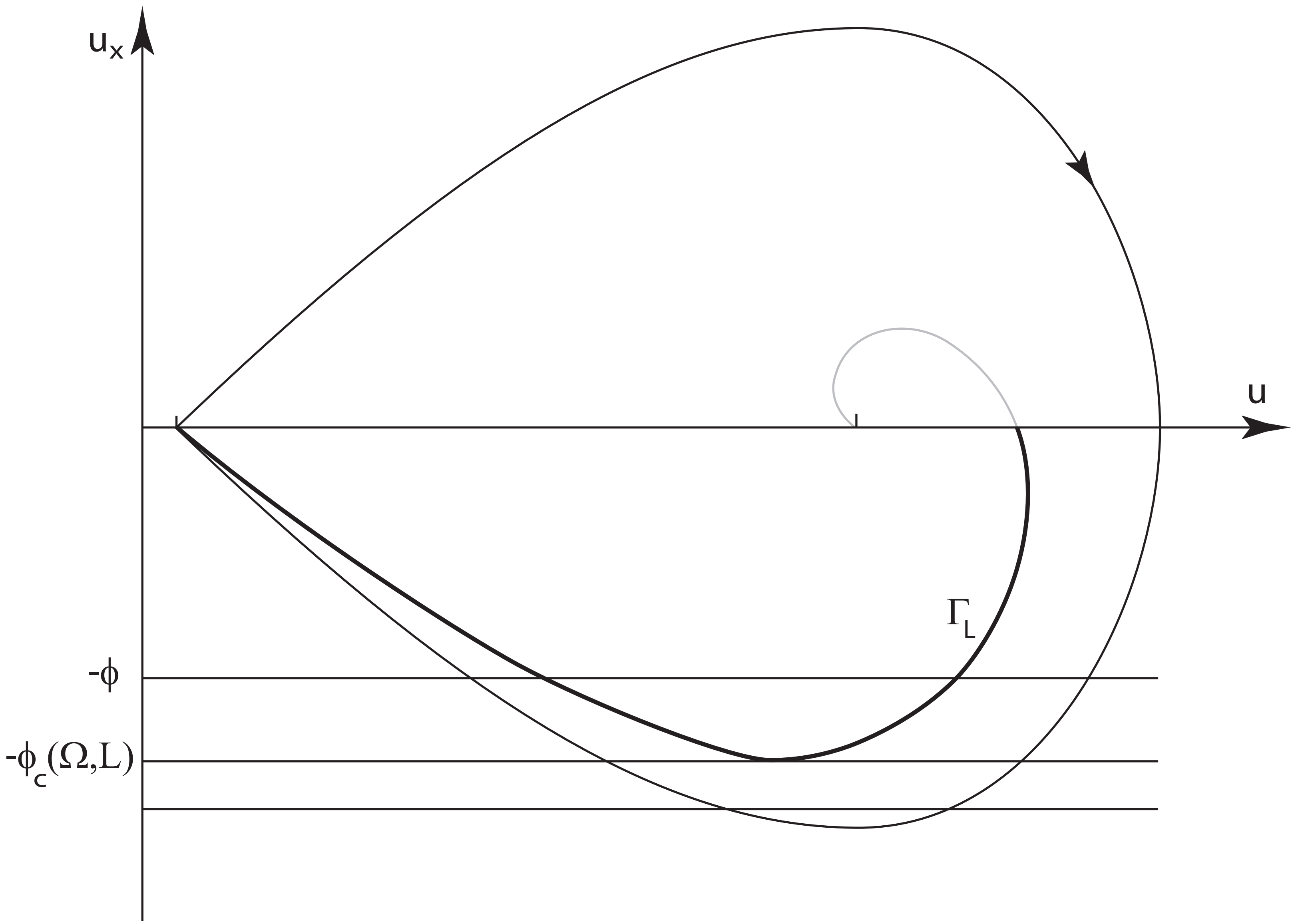

u
u_c+2π
u_c
0
x


u
x

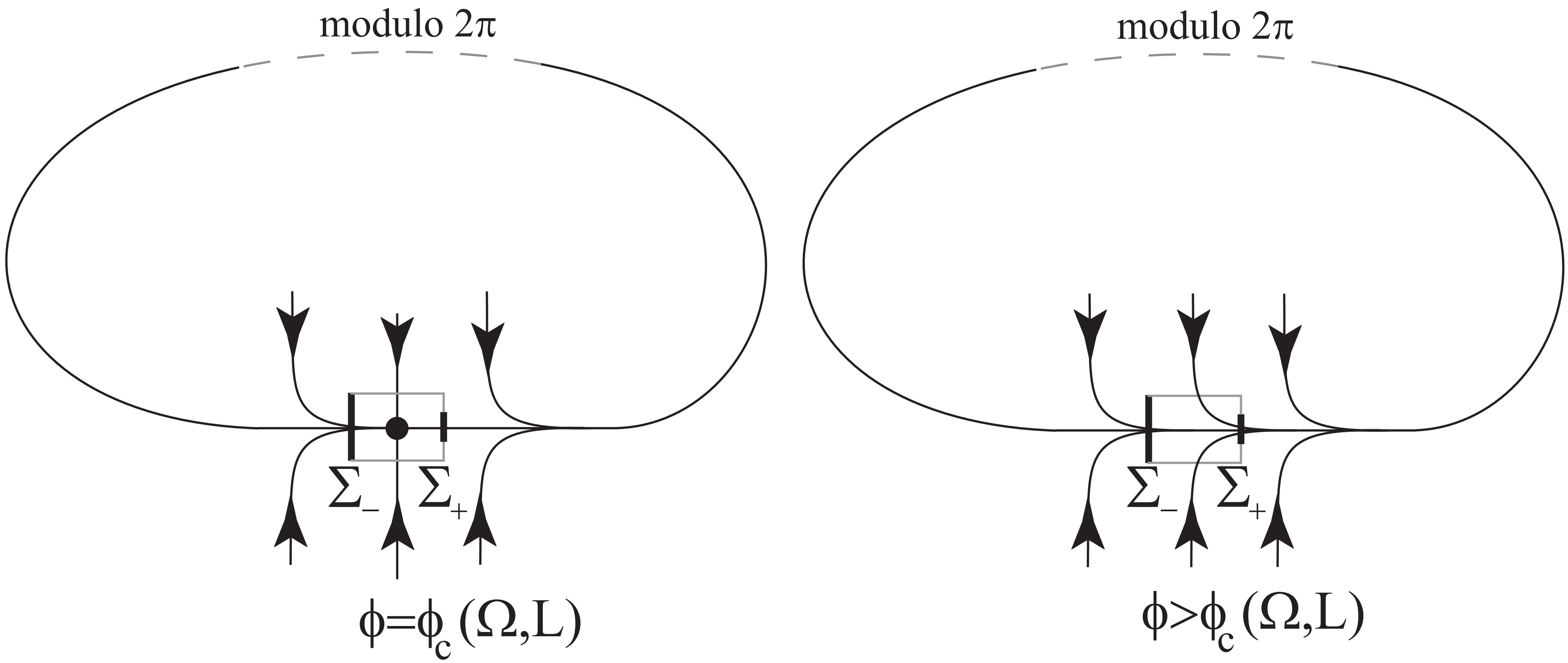
II.3.Domaine spatial borné

$$x \in [0, L],\ t \geq 0.$$

$$\begin{cases} u_t = -\sin u + \Omega + u_{xx} \\ u_x(x = 0, t) = -\phi \\ u_x(x = L, t) = 0 \end{cases}$$



u_x
u
Γ_L
$-\phi$
$-\phi_c(\Omega,L)$

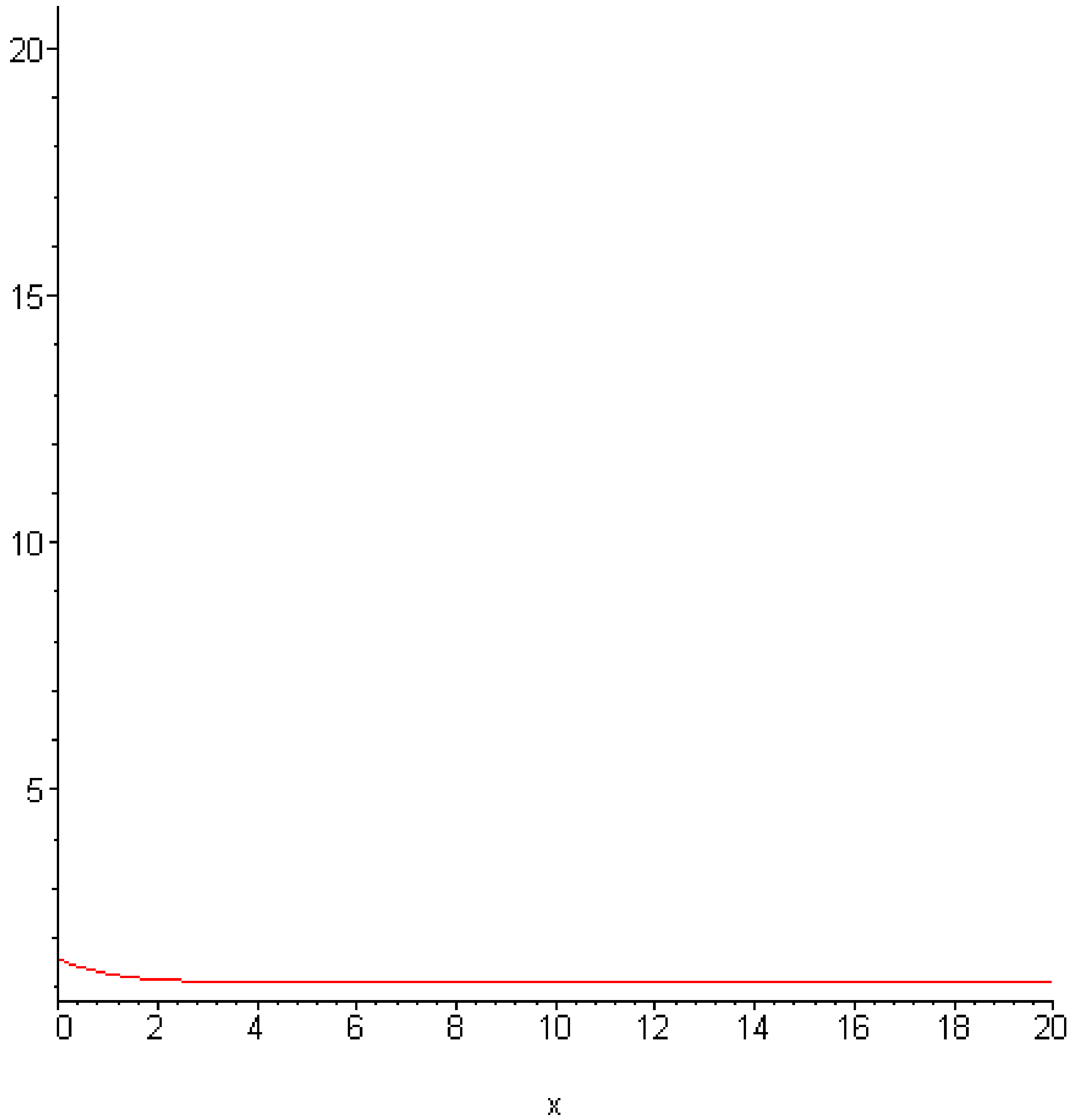

modulo 2π
Σ₋
Σ₊
ϕ=ϕc (Ω,L)
modulo 2π
Σ₋
Σ₊
ϕ>ϕc (Ω,L)

Théorème *Il existe* $\varepsilon > 0$ *tel que si* $\phi_c(\Omega, L) < \phi < \phi_c(\Omega, L) + \varepsilon$*, il existe une orbite périodique (modulo* 2π*) globalement attractive. Sa période est de l'ordre de* $(\phi - \phi_c(\Omega, L))^{-1/2}$.

$L = 20$

$\Omega = 0.9$

$\phi = 0.4$

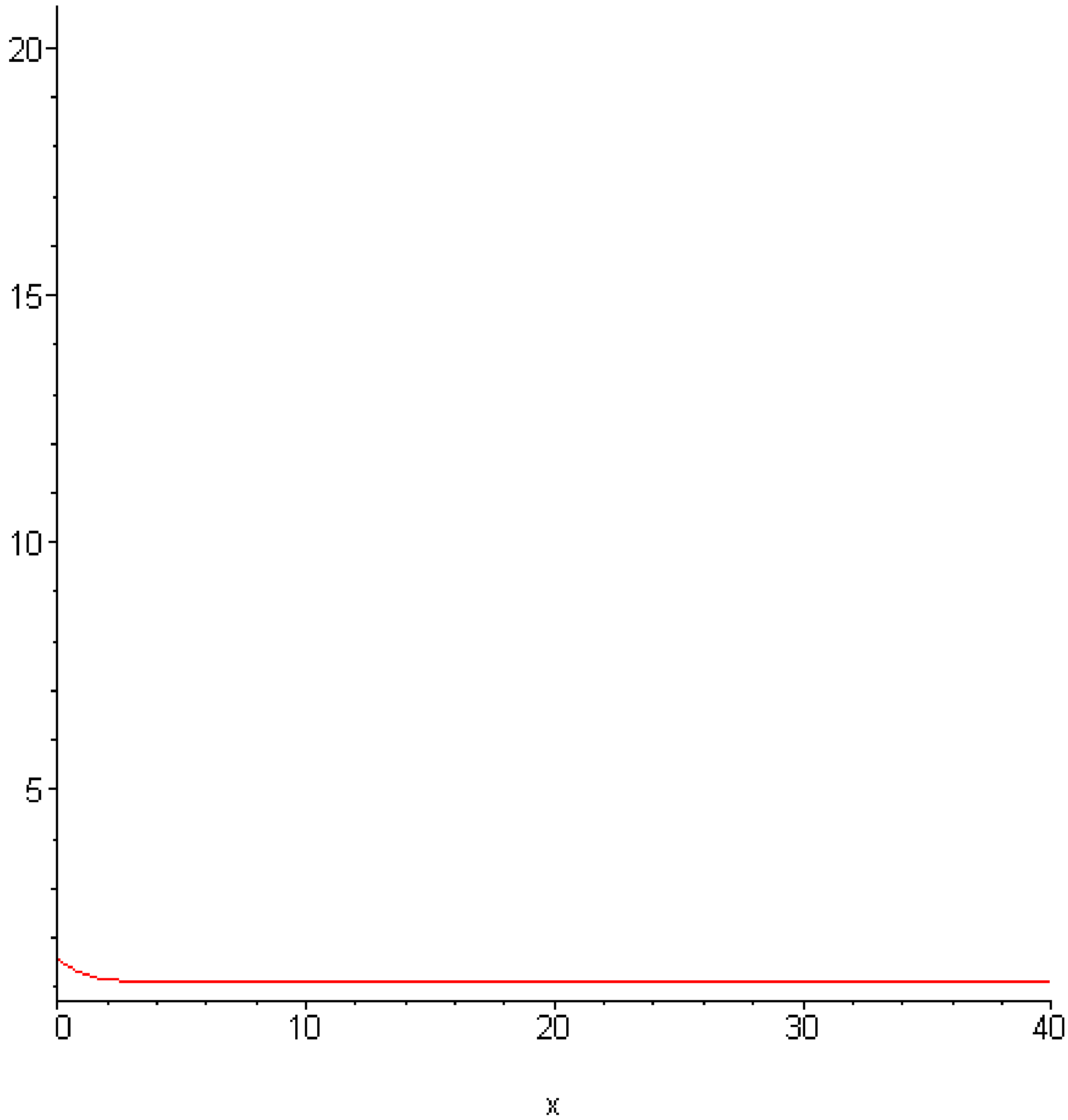


$L = 20$

$\Omega = 0.9$

$\phi = 0.4$

$$L = 40$$
$$\Omega = 0.9$$
$$\phi = 0.4$$

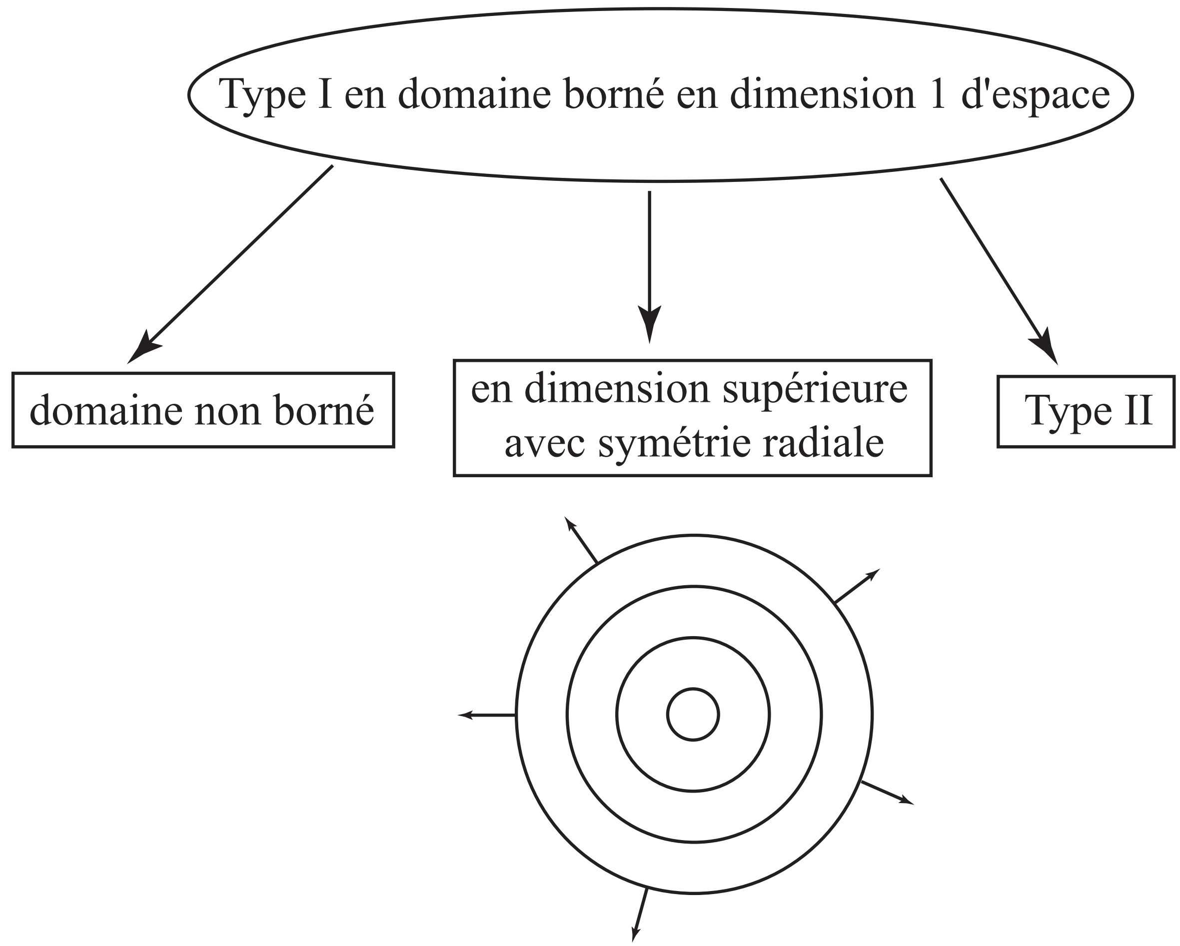


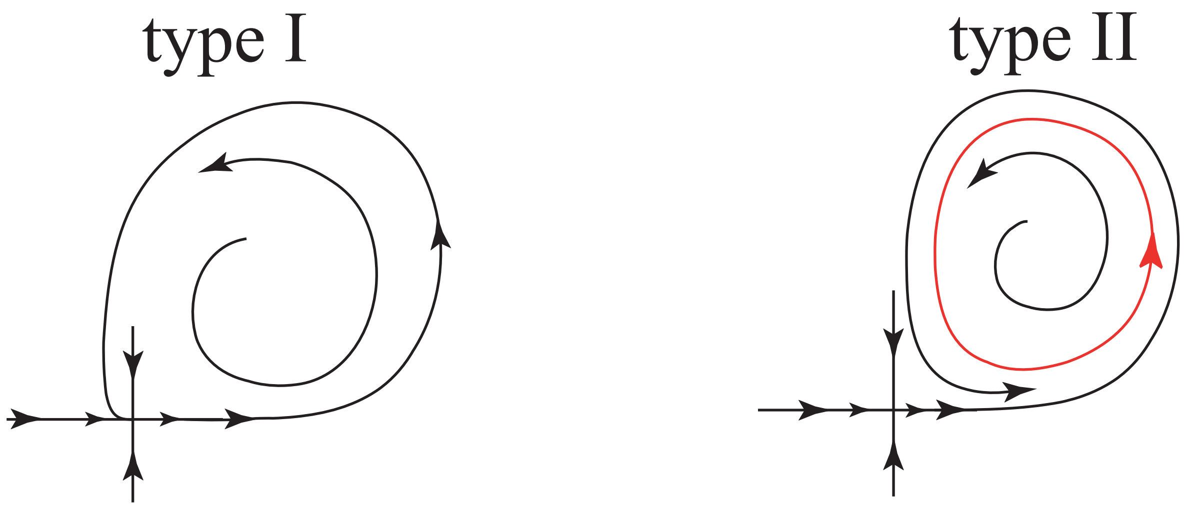
$L = 40$

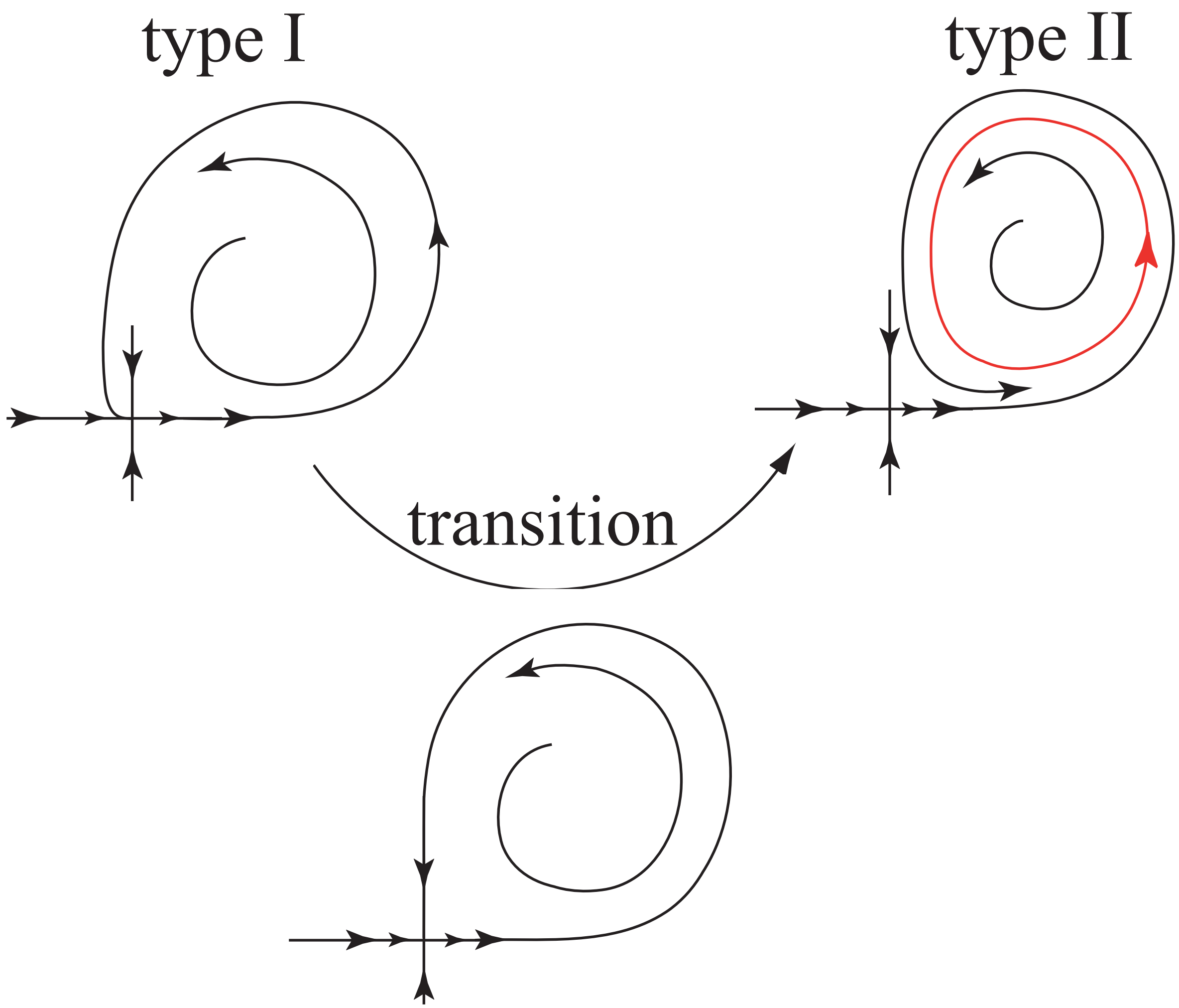
$\Omega = 0.9$

$\phi = 0.4$

II.4.Perspectives




type I
type II



avec de l'inertie $\rightarrow$ $\alpha u_{tt} + u_t + \sin u - \Omega + u_{xx} = 0$